Theorem 1 *Given a system of m equations in n unknowns, let B be the $m \times (n+1)$ augmented matrix. Recall r is the number of leading ones in the REF of B, also the number of parameters in a solution is $n - r$.*

- *If $r = n$, there is a unique solution (no parameters in the solution).*
- *If $r > n$ (so $r = n + 1$) the system is inconsistent (no solution).*
- *If $r < n$, either the system is inconsistent (no solution) or an $n - r$-parameter solution.*
  - *In this case, the difference is determined only by the values of the constants (the b_i).*

Homogeneous Systems

Given a system of m equations in n unknowns

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{array}$$

If all of the constant terms are zero, i.e. $b_i = 0$ for $i = 1, \ldots m$ the corresponding system of equations is called a *homogeneous system system of equations*.

Example 2

$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & - & 3x_3 & + & x_4 & = & 0 \\ & & x_2 & + & x_3 & - & 3x_4 & = & 0 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \end{array}$$

A homogeneous system of equations **always** has the solution

$$x_1 = x_2 = \ldots = x_n = 0$$

This is called the *Trivial Solution*.

Since a homogeneous system always has a solution (the trivial solution), it can never be inconsistent.

Thus a homogeneous system of equations always either has a unique solution or an infinite number of solutions.

Theorem 3 *If $n > m$ then a homogeneous system of equations has infinitely many solutions.*

Example 4

1.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ x_1 & + & x_2 & + & 2x_3 & = & 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right) \quad \begin{array}{rcl} R_2 & \to & R_2 - R_1 \\ R_3 & \to & R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Write back:

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

So the trivial solution $(x_1, x_2, x_3) = (0, 0, 0)$ is the only solution.

2.

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & 2x_3 & = & 0 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 3 & 2 & 0 \end{array}\right) \quad \begin{array}{ccc} R_2 & \to & R_2 - R_1 \\ R_3 & \to & R_3 - 2R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad R_3 \to R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Write back:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

Which has the 1-parameter solution:
Let $t \in \mathbb{R}$, $x_3 = t$, $x_2 = -t$, $x_1 = 0$.
Or $(x_1, x_2, x_3) = (0, -t, t)$.