Eigenvalues

Definition 1 *Given an* $n \times n$ *matrix* A*, a scalar* $\lambda \in \mathbb{C}$*, and a non zero vector* $\mathbf{v} \in \mathbb{R}^n$ *we say that* λ *is an* eigenvalue *of* A*, with corresponding* eigenvalue $\mathbf{v}$ *if*

$$A\mathbf{v} = \lambda\mathbf{v}$$

Notes

- Eigenvalues and eigenvectors are only defined for square matrices.
- Even if A only has real entries we allow for the possibility that λ and $\mathbf{v}$ are complex.
- Surprisingly, a given square $n \times n$ matrix A, admits only a few eigenvalues (at most n), but infinitely many eignevectors.

Given A, we want to find possible values of λ and $\mathbf{v}$.

Example 2

1.

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So 2 is an eigenvalue of $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ with corresponding eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

2. Is $\mathbf{u} = (1, 0, 1)$ an eigenvector for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix}?$$

$$A\mathbf{u} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$(1, 0, 1) = \lambda(1, 2, 1)$ has no solution for λ, so $(1, 0, 1)$ is not an eigenvector of A.

Eigenspaces

Theorem 3 *If* $\mathbf{v}$ *is an eigenvector, corresponding to the eigenvalue* λ_0 *then* $c\mathbf{u}$ *is also an eigenvector corresponding to the eigenvalue* λ_0*.*

If $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are an eigenvectors, both corresponding to the eigenvalue* λ_0*, then* $\mathbf{v}_1 + \mathbf{v}_2$ *is also an eigenvector corresponding to the eigenvalue* λ_0*.*

Proof:

$$A(c\mathbf{v}) = cA\mathbf{v} = c\lambda_0\mathbf{v} = \lambda_0(c\mathbf{v})$$

$$A(\mathbf{v}_1 + \mathbf{v}_2) = A\mathbf{v}_1 + A\mathbf{v}_2 = \lambda_0\mathbf{v}_1 + \lambda_0\mathbf{v}_2 = \lambda_0(\mathbf{v}_1 + \mathbf{v}_2)$$

Corollary 4 *The set of vectors corresponding to an eigenvalue* λ_0 *of an* $n \times n$ *matrix is a subspace of* $\mathbb{R}^n$*.*

Thus when we are asked to find the eigenvectors corresponding to a given eigenvalue we are being asked to find a subspace of $\mathbb{R}^n$. Generally, we describe such a space by giving a basis for it.

Definition 5 *Given an $n \times n$ square matrix A, with an eigenvalue λ_0, the set of eigenvectors corresponding to λ_0 is called the* Eigenspace *corresponding to λ.*

$$E_{\lambda_0} = \{\mathbf{v} \in \mathbb{R}^n \mid A\mathbf{v} = \lambda_0\mathbf{v}\}$$

The dimension of the eigenspace corresponding to λ, dim(E_{λ_0}), is called the Geometric Multiplicity *of the Eigenvalue λ_0*

Characteristic Polynomials

Given square matrix A we wish to find possible eigenvalues of A.

$$\begin{array}{rrcl} & A\mathbf{v} & = & \lambda\mathbf{v} \\ \Rightarrow & \mathbf{0} & = & \lambda\mathbf{v} - A\mathbf{v} \\ \Rightarrow & \mathbf{0} & = & \lambda I\mathbf{v} - A\mathbf{v} \\ \Rightarrow & \mathbf{0} & = & (\lambda I - A)\mathbf{v} \end{array}$$

This homogeneous system will have non trivial solutions if and only if $\det(\lambda I - A) = 0$.

Example 6

Find all eigenvalues of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{array}{rcl} \det(\lambda I - A) & = & \det\left(\lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) \\ & = & \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} \\ & = & (\lambda - 1)^2 - 1 \\ & = & \lambda^2 - 2\lambda \\ & = & \lambda(\lambda - 2) \end{array}$$

So $\det(\lambda I - A) = 0$ if and only if $\lambda = 0$ or 2.

Definition 7 *Given an $n \times n$ square matrix A, det($\lambda I - A$) is a polynomial in λ of degree n. This polynomial is called the* characteristic polynomial *of A, denoted $p_A(\lambda)$.*

The equation $p_A(\lambda) = 0$ is called the characteristic equation *of A.*

Example 8

Find the characteristic polynomial and the characteristic equation of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} p_A(\lambda) &= \det(\lambda I - A) \\ &= \begin{vmatrix} \lambda - 1 & -2 & -1 \\ 0 & \lambda - 2 & 1 \\ 0 & 0 & \lambda - 1 \end{vmatrix} \\ &= (\lambda - 1)(\lambda - 2)(\lambda - 1) \\ &= (\lambda - 1)^2(\lambda - 2) \end{aligned}$$

If A is an $n \times n$ matrix, $p_A(\lambda)$ will be a degree n polynomial. By the Fundamental Theorem of Algebra, a degree n polynomial has n roots over the complex numbers. Note that some of these roots may be equal.

Given an $n \times n$ square matrix A, with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then

$$p_A(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \ldots (\lambda - \lambda_k)^{r_k}$$

for some positive integers $r_1, r_2, \ldots, r_k$.

Definition 9 *For a given i the positive integer r_i is called the* algebraic multiplicity *of the eigenvalue λ_i.*

Theorem 10 *For a given square matrix the geometric multiplicity of any eigenvalue is always less than or equal to the algebraic multiplicity.*

Example 11

Find the distinct eigenvalues of A above and give the algebaraic multiplicity in each case.

$$p_A(\lambda) = (\lambda - 1)^2(\lambda - 2)$$

So the distinct eigenvalues of A are $\lambda = 1$ and $\lambda = 2$.

The algebraic multiplicity of $\lambda = 1$ is 2.
The algebraic multiplicity of $\lambda = 2$ is 1.

Finding Egenvalues & Eigenspaces

Note if we are asked to find eigenvectors of a matrix A, we are actually being asked to find eigenspaces.

Given an $n \times n$ matrix A we may find the Eigenvalues and Eigenspaces of A as follows:

1. Find the characteristic polynomial

$$p_A(\lambda) = |\lambda I - A|.$$

2. Find the roots of $p_A(\lambda) = 0$, $\lambda_1, \ldots, \ldots \lambda_k$, n roots with possible repetition.

$$p_A(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \ldots (\lambda - \lambda_k)^{r_k}$$

3. For each distinct eigenvalue λ_i in turn solve the homogeneous system

$$(\lambda_i I - A)\mathbf{x} = \mathbf{0}$$

The solution set is E_{λ_i}.

Example 12

1. Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$p_A(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 2)$$

Solutions are $\lambda = 0$ or 2.

$\underline{\lambda = 0}$ Solving $-A\mathbf{x} = \mathbf{0}$.

$$\left(\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right) \quad R_2 \to R_2 - R_1 \qquad \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$, $x_2 = t$, $x_1 = -t$, so

$$E_0 = \{t(-1, 1) \mid t \in \mathbb{R}\}.$$

$\underline{\lambda = 2}$ Solving $(2I - A)\mathbf{x} = \mathbf{0}$.

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right) \quad R_2 \to R_2 + R_1 \qquad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$, $x_2 = t$, $x_1 = t$, so

$$E_2 = \{t(1, 1) \mid t \in \mathbb{R}\}.$$

In both cases Algebraic Multiplicity = Geometric Multiplicity = 1

2. Find all eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

We proved earlier that

$$p_A(\lambda) \;=\; (\lambda - 1)^2(\lambda - 2)$$

So eigenvalues are $\lambda = 1$ and $\lambda = 2$.

$\lambda = 1$ has algebraic multiplicity 2, while $\lambda = 2$ has algebraic multiplicity 1.

$\underline{\lambda = 1}$ Solving $(I - A)\mathbf{x} = \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \;-\; \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 0 & -2 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \;\rightarrow\; \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$, set $x_1 = t$ and $x_3 = 0$, $x_2 = 0$.

$$E_1 = \{t(1,0,0) \mid t \in \mathbb{R}\}$$

geometric mutiplicity $= 1 \neq$ algebraic multiplicity.

$\underline{\lambda = 2}$ Solving $(2I - A)\mathbf{x} = \mathbf{0}$.

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) - \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \qquad R_3 \to R_3 - R2$$

$$\left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$, set $x_2 = t$ and $x_3 = 0$, $x_1 = 2t$.

$$E_2 = \{t(2,1,0) \mid t \in \mathbb{R}\}$$

Theorem 13 *If A is a square triangular matrix, then the eigenvalues of A are the diagonal entries of A*

Theorem 14 *A square matrix A is invertible if and only if $\lambda = 0$ is* **not** *an eigenvalue of A.*

Theorem 15 *Given a square matrix A, with an eigenvalue λ_0 and correspoonding eigenvector $\mathbf{v}$, then for any positive integer k, λ_0^k is an eigenvalue of A^k and $A^k\mathbf{v} = \lambda_0^k\mathbf{v}$.*

Theorem 16 *If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of a matrix, and $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are corresponding eigenvectors respectively, then the are linearly independent. i.e. Eigenvectors corresponding to distinct eigenvalues are linearly independent.*

Corollary 17 *If A is an $n \times n$ matrix, A has n linearly independent eigenvectors if and only if the algebraic multiplicity = geometric multiplicity for each distinct eigenvalue.*