eg Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Consider $A\underline{v} = \lambda \underline{v}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\cancel{\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - (}$$

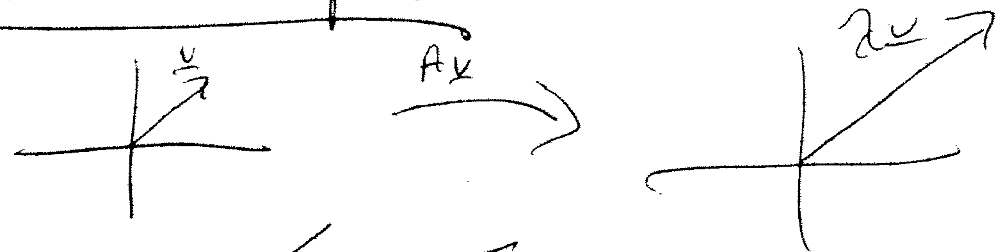
$$\Rightarrow \left(\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \underline{0}$$

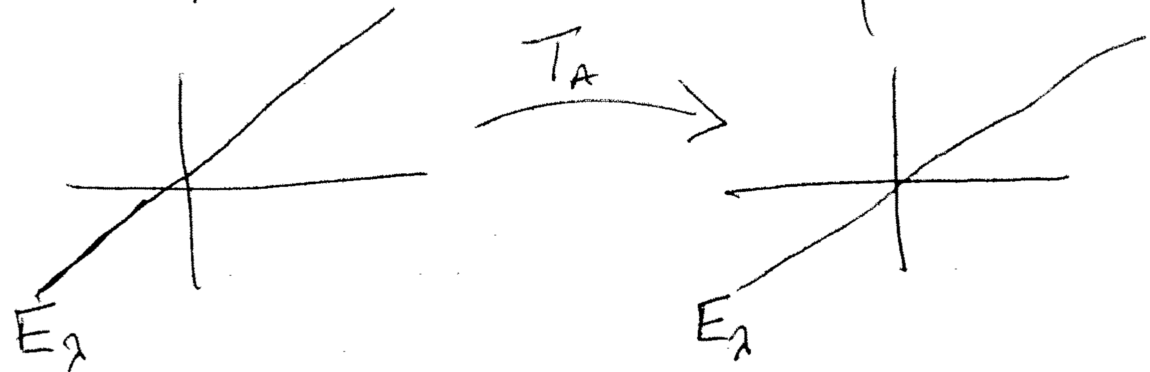
$$\Rightarrow \left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvalues & Eigenvectors

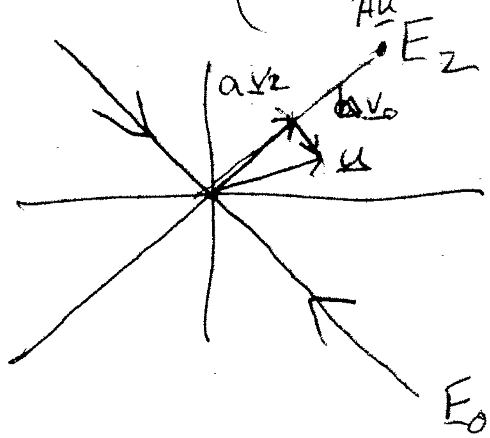
Geometric Interpretatⁿ





eg $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\lambda = 0, 2$. $E_0 = \{ t(-1,1) \mid t \in \mathbb{R} \}$

$E_2 = \{ t(1,1) \mid t \in \mathbb{R} \}$



Everything on the line E_0 gets mapped to $\underline{0}$ by A.
Everything on E_2 gets mapped to E_2

$\underline{u} = a\underline{v}_2 + b\underline{v}_0$ $\underline{v}_2 = (1,1)$ $\underline{v}_0 = (-1,1)$

$A\underline{u} = Aa\underline{v}_2 + Ab\underline{v}_0 = 2a\underline{v}_2$

$A\underline{v}_2 = 2\underline{v}_2$ $A\underline{v}_0 = 0$

Given An $n \times n$ mx A, if we can find a basis of $\mathbb{R}^n$ consisting of Eigenvectors of A, then we can express any $\underline{u} \in \mathbb{R}^n$ as $\underline{u} = a_1\underline{v}_1 + a_2\underline{v}_2 + \ldots + a_n\underline{v}_n$ where $\underline{v}_i$ is an eigenvector corresponding to λ_i

No $A\underline{u} = a_1 A\underline{v}_1 + a_2 A\underline{v}_2 + \ldots + a_n A\underline{v}_n = a_1\lambda_1\underline{v}_1 + a_2\lambda_2\underline{v}_2 + \ldots + a_n\lambda_n\underline{v}_n$

eg 1) Let $A = \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix}$ Find the eigenvalues & e-vectors of A.

$$P_A(\lambda) = |\lambda I - A| = \left| \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ 0 & -3 \end{pmatrix} \right|$$

$$= \begin{vmatrix} \lambda + 2 & -1 \\ 0 & \lambda + 3 \end{vmatrix}$$

$$= (\lambda + 2)(\lambda + 3)$$

$P_A(\lambda) = 0 \iff \lambda = -2, -3$ ← Eigenvalues

$\underline{\lambda = -2}$ $\quad \lambda I - A = -2I - A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$

Solve $\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow[R_2 \to R_2 + R_1]{} \left(\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Let $t \in \mathbb{R}$, $x_1 = t$, $x_2 = 0$

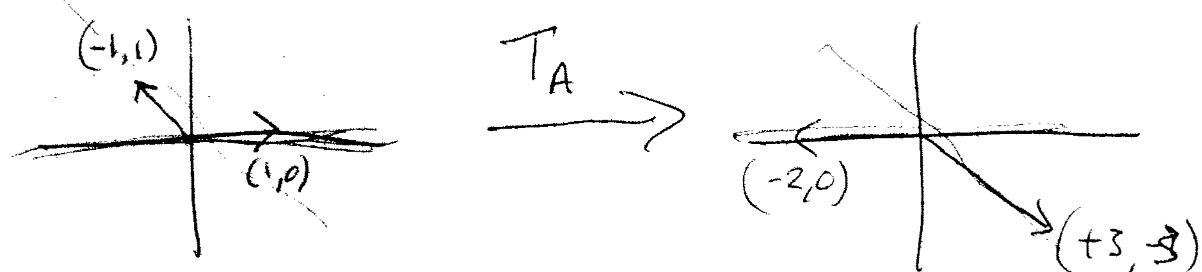
Sol^{n}s of the form $t(1, 0)$ $\qquad \underline{(1, 0)}$

$$E_{-2} = \{ t(1,0) \mid t \in \mathbb{R} \}$$

$\underline{\lambda = -3}$ $\quad \lambda I - A = -3I - A = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$

Let $t \in \mathbb{R}$ $x_2 = t$, $x_1 = -t$ $\qquad t(-1, 1)$

$$E_{-3} = \{ t(-1,1) \mid t \in \mathbb{R} \}$$



eg $\underline{u} = (0, 1)$ $\qquad \underline{u} = \underbrace{(1,0)}_{\underline{v}_{-2}} + \underbrace{(-1,1)}_{\underline{v}_{-3}}$

So $T_A(\underline{u}) \overset{= A\underline{u}}{=} (-2, 0) + (-3, 3) = (-1, 3)$

2) Let $A = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ Find e.val & e.vec for A 3

$$P_A(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -3 & 2 \\ 0 & \lambda-1 & -4 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)^2(\lambda-2)$$

[Note: Algebraic Mult. of $\lambda=1$ is 2
" " $\lambda=2$ is 1]

Eigenvalues are $\lambda=1$ & $\lambda=2$.

$\underline{\lambda=1}$ Solve $(I-A)x=0$ $\quad I-A = \begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & -1 \end{pmatrix}$

Solve $\left(\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$ $R_2 \to \frac{-1}{4}R$

$\left(\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$ $R_3 \to R_3 + R_2$

$\left(\begin{array}{ccc|c} 0 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_3 = 0 \quad x_2 = 0$
Let $t \in \mathbb{R}$ $\quad x_1 = t$.
$(1,0,0)$

$E_1 = \{t(1,0,0) \mid t \in \mathbb{R}\}$

Not geometric mult. is 1

$\underline{\lambda=2}$ $2I - A = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$

Solve $\left(\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Let $t \in \mathbb{R}$ $x_3 = t$
$x_2 = 4t$, $x_1 = 3(4t) - 2t$
$= 10t$.

$E_2 = \{t(10,4,1) \mid t \in \mathbb{R}\}$
g.m $= 1$.

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3) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ Find e.val. & e.vec.

$P_A(\lambda) = \begin{vmatrix} \lambda-1 & +1 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-1)^2+1 = \lambda^2-2\lambda+1+1$

$= \lambda^2 - 2\lambda + 2$ So $\lambda = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm \frac{\sqrt{-4}}{2} = 1 \pm i$

So Eigenvalues are $1+i$ & $1-i$

We can still solve for e.vectors, but they will be complex.

$\lambda = 1+i$ Solve $((1+i)I - A)\underline{x} = 0$

$\left(\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right)$ $R_1 \to -iR_1$

$\left(\begin{array}{cc|c} 1 & -i & 0 \\ -1 & i & 0 \end{array}\right)$ $R_2 \to R_2 + R_1$

$\left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$ let $t \in \mathbb{R}$ $x_2 = t$, $x_1 = it$

$E_{1+i} = \{ t(i, 1) \mid t \in \mathbb{R} \}$

$\lambda = 1-i$ Ex

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4) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Find e.val. & e.vect. of A.

$P_A(\lambda) = (\lambda-1)^2(\lambda-2)$

eigenvalues $\lambda = 1$ & $\lambda = 2$.

$\left[\begin{array}{l} \text{a.m. of } \lambda=1 \text{ is } 2 \\ \text{〃 〃 } \lambda=2 \text{ 〃 } 1 \end{array}\right]$

$\underline{\lambda = 1}$ $\lambda I - A = I - A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

Solve $(\lambda I - A)\underline{x} = \underline{0}$

$\left(\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $R_2 \to R_2 - R_1$

$\left(\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Let $s, t \in \mathbb{R}$ $x_1 = s$, $x_3 = t$, $x_2 = -t$

$E_1 = \{ s(1,0,0) + t(0,-1,1) \mid s,t \in \mathbb{R} \}$.

g.m. of $\lambda = 1$ is 2.

$\underline{\lambda = 2}$ $2I - A$ $\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$ $R_3 \to R_3 - R_2$

$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ let $t \in \mathbb{R}$ $x_2 = t$, $x_3 = 0$

$x_1 = t$

$E_2 = \{ t(1,1,0) \mid t \in \mathbb{R} \}$

3 eigenvectors that span $\mathbb{R}^3$ $(1,0,0)$, $(0,-1,1)$, $(1,1,0)$