Complex Numbers

<u>Oct 5 09</u>

eg of complex #'s $3+2i$, $1+i$, $-1+i$, $1-i$

$\pi+3i$, $1+\sqrt{2}i$, $\pi+\sqrt{2}i$, $\pi-\sqrt{2}i$

Imaginary part can be 0

$1, -1, 3, \pi, \sqrt{2}, \ldots$ $\mathbb{R} \subseteq \mathbb{C}$

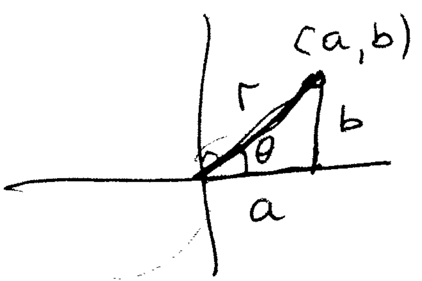
real #'s are contained in the complex #'s

Could have Real part 0.

i, $-i$ $3i$, πi, $-\pi i$, $\sqrt{2}i$

- called Pure imaginary #'s.

There is a natural 1-1 correspondence between $\mathbb{C}$ & $\mathbb{R}^2$



$z = a+ib$

$r = |z| = \sqrt{a^2+b^2}$ - modulus of z

$\theta = \arg(z)$ - argument of z.
angle with +ve x-axis.

$\theta = [0, 2\pi]$ or $[-\pi, \pi]$ or whatever

In fact $\arg(z)$ is multivalued — if $\arg(z)=\theta$ then $\arg(z) = \theta + 2\pi n$ for any integer n.

$\operatorname{Arg}(z)$ is the value in $[0, 2\pi]$

$z = a+ib$ is called rectangular form.

$z = r\cos\theta + i r\sin\theta = re^{i\theta}$ is called polar form.

<u>DeMoivres Thm</u> $r(\cos\theta + i\sin\theta) = re^{i\theta}$

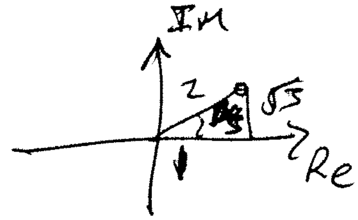
Converting ~~from~~ between Polar ~~to~~ & Rectangular forms

$z = a + ib \qquad a = r\cos\theta, \quad b = r\sin\theta$

eg i) Find $z = 2e^{i\pi/3}$ in rectangular form.

$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad [r = 2, \ \theta = \frac{\pi}{3}]$

$a = r\cos\theta = 1 \qquad b = r\sin\theta = \sqrt{3}$

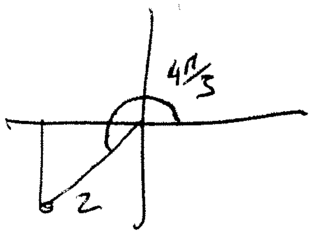


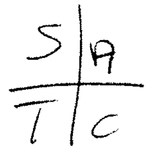
So $2e^{i\pi/3} = 1 + \sqrt{3}\,i$

2) $z = 2e^{\frac{4\pi}{3}i}$

$\cos\frac{4\pi}{3} = -\frac{1}{2}$

$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$





So $2e^{\frac{4\pi i}{3}} = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

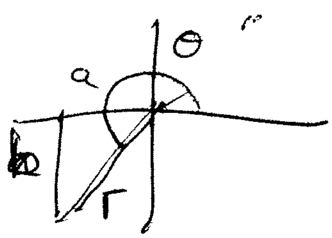
$= 2\left(-\frac{1}{2} + -\frac{\sqrt{3}}{2}i\right)$

$= -1 - \sqrt{3}\,i$

To go the other way $z = a + ib$

Use $r = |z| = \sqrt{a^2+b^2} \qquad \theta = \tan^{-1}\frac{b}{a}$ * $b > 0$

$= \tan^{-1}\frac{b}{a} + \pi \quad b < 0$



eg i) $1 + i \qquad r = \sqrt{1^2 + 1^2} = \sqrt{2} \qquad \tan\theta = \frac{1}{1} = 1$

$1 + i = \sqrt{2}\,e^{i\pi/4} \qquad \theta = \tan^{-1} 1 = \frac{\pi}{4}$

2) $1 - \sqrt{3}\,i \ \boxed{= 2e^{\frac{5\pi}{3}i}}$

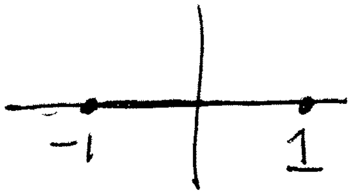
$r = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2 \qquad \theta = \tan\frac{-\sqrt{3}}{1} + \pi = \frac{2\pi}{3} + \pi$

$= \frac{5\pi}{3}$

eg 1) Find $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{57}$

$$\frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{\frac{i\pi 5}{3}}$$

$$\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{57} = \left(e^{\frac{i5\pi}{3}}\right)^{57} = e^{\frac{57 i\pi 5}{3}} = e^{19\cdot 5\pi i} = -1$$



$\arg(1) = 0, 2\pi, 4\pi, \ldots$

$\arg(-1) = \pi, 3\pi, 5\pi, \ldots$

Euler's Formula: $e^{i\pi} + 1 = 0$

or $e^{i\pi} = -1$

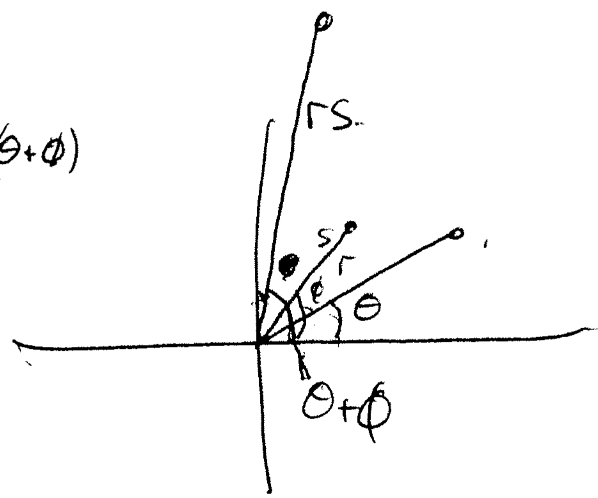
Adding is much easier in rectangular form

Generally mult. is easier in polar form.

$z = re^{i\theta}$ & $\omega = se^{i\phi}$

$= a+ib$ $\quad = c+id$

$z\omega = (re^{i\theta})(se^{i\phi}) = rs\, e^{i(\theta+\phi)}$



Division $\quad \frac{1}{z} = \frac{1}{|z|^2}\bar{z}$

$z = re^{i\theta} \qquad \bar{z} = re^{-i\theta}$

$z\bar{z} = r^2 \Rightarrow \frac{\bar{z}}{r^2} = \frac{1}{z}$

$$\begin{aligned} z\cdot\bar{z} &= re^{i\theta}\cdot re^{-i\theta} \\ &= r^2 e^{i(\theta-\theta)} \\ &= r^2 e^0 \\ &= r^2 \end{aligned}$$

$$\frac{1}{z} = \frac{1}{|z|}\bar{z}$$

eg Find z s.t. $2+i = (3+4i)z$

$$z = \frac{2+i}{3+4i} = 2+i\left(\frac{1}{3+4i}\right)$$

Now $$\frac{1}{3+4i} = \frac{1}{|3+4i|^2}(3-4i) \qquad |3+4i| = \sqrt{3^2+4^2}$$

$$= \frac{1}{9+16}(3-4i)$$

$$= \frac{1}{25}(3-4i)$$

$$z = (2+i)\frac{1}{25}(3-4i)$$

$$= \frac{1}{25}(2+i)(3-4i)$$

$$= \frac{1}{25}(6-8i+3i-4i^2) \qquad -4i^2 = -4(-1) = 4$$

$$= \frac{1}{25}(10-5i)$$

$$= \frac{10}{25} - \frac{5}{25}i = \frac{2}{5} - \frac{1}{5}i$$

$$\boxed{= \frac{1}{5}(2-i)}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

I

Transpose

eg 1) $\begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 2 & 1 \\ 5 & 1 & 2 \end{pmatrix}$

2) $A = [a_{ij}] = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 5 & 1 & 2 \end{pmatrix}$

$a_{12} = 3 \qquad a_{32} = 1 \qquad a_{31} = 5 \qquad a_{33} = 2$

$a_{23} = 1$

Matrix Mult^{n}

eg 1) Find AB where $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 8 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

2) Find $A^T B$ where $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$ & $B = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$A^T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

$$A^T B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3+1 & 1 & 1 \\ 3+1 & 0 & 1+1 \\ 6+1+2 & 2 & 2+1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 1 & 1 \\ 4 & 0 & 2 \\ 9 & 2 & 3 \end{pmatrix}$$

3) Find $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2+1+2 & 2+2 \\ 1+1 & 2+1 \\ 1+2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 2 & 3 \\ 3 & 2 \end{pmatrix}$

4) Find $\begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 4 \end{pmatrix}$ Not Possible

5) $\begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 4 \end{pmatrix}^T$

$$\begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 13 \\ 4 & 7 \\ 3 & 3 \end{pmatrix}$$

6) Find $\begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 \\ 1 & 4 \end{pmatrix}$