eg Find AB where

$$A = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 16 & 5 & 2 & 0 \\ 0 & 5 & 10 & 3 \end{pmatrix}$$

$|A| = 1 \times 2 \times 3 \times 1 = 6 \qquad |B| = 3 \times 1 \times 2 \times 3 = 18$

$|AB| = |A||B| = 6 \times 18 = \underline{108}$

$$AB = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 16 & 5 & 2 & 0 \\ 0 & 5 & 10 & 3 \end{pmatrix} = \begin{pmatrix} 82 & 48 & 58 & 15 \\ \cancel{1546} 177 & 78 & 18 \\ 48 & 40 & 56 & 15 \\ 0 & 5 & 10 & 3 \end{pmatrix}$$

$160 - 9 = 151$
$+6$

$$\begin{array}{c} i \\ j \end{array}\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \end{vmatrix} \qquad R_j \to R_j - R_i$$

$$\begin{array}{c} i \\ j \end{array}\begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \end{vmatrix} = 0$$

eg 1) a) Find $\underline{u} \times \underline{v}$ where

$\underline{u} = (1,2,1)$ & $\underline{v} = (0,2,1)$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix} = (2-2)\underline{i} - \underline{j}(1-0) + \underline{k}(2-0)$$

$$= -\underline{j} + 2\underline{k} = (0,-1,2)$$

Check: $\underline{u} \cdot (\underline{u} \times \underline{v}) = (1,2,1) \cdot (0,-1,2) = 0 + -2 + 2 = 0$

$\underline{v} \cdot (\underline{u} \times \underline{v}) = (0,2,1) \cdot (0,-1,2) = 0$

b) Find the standard eq^{n} of the plane ~~eo~~ // to the vectors $\underline{u}$ & $\underline{v}$ through O.

Take $\underline{n} = \underline{u} \times \underline{v} = (0,-1,2)$ (from part a)

So $\underline{-y + 2z = 0}$ (d = 0 since through O)

2) Let $\underline{w} = (1,1,1)$ ($\underline{u}$ & $\underline{v}$ as above)

a) Find $(\underline{u} \times \underline{v}) \cdot \underline{w} = (0,-1,2) \cdot (1,1,1) = 0 - 1 + 2 = 1$

b) Find $\underline{u} \cdot (\underline{v} \times \underline{w}) = \cancel{(1,1,1) \cdot (0,-1,2) = 0 - 1 + 2 = 1}$

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \underline{i}(2-1) - \underline{j}(0-1) + \underline{k}(0-2)$$

$$= \underline{i} + \underline{j} - 2\underline{k}$$

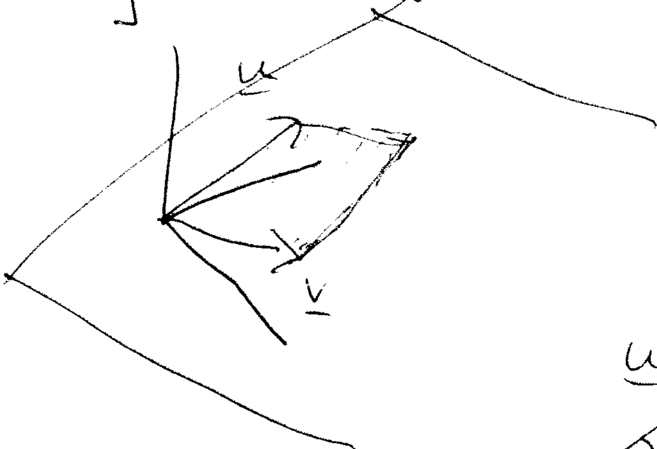
So $\underline{v} \times \underline{w} = (1,1,-2)$

$\underline{u} \cdot (\underline{v} \times \underline{w}) = (1,2,1) \cdot (1,1,-2) = 1 + 2 - 2 = 1$

So $(\underline{u} \times \underline{v}) \cdot \underline{w} = \underline{u} \cdot (\underline{v} \times \underline{w})$.

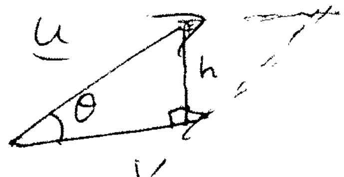
Area of a Parallelogram in $\mathbb{R}^3$

Any 2 non $\parallel$ vectors in $\mathbb{R}^3$ map out a $\parallel$gram.



What is the area of this $\parallel$gram?

= base $\times$ height.



base $= \|\underline{v}\|$

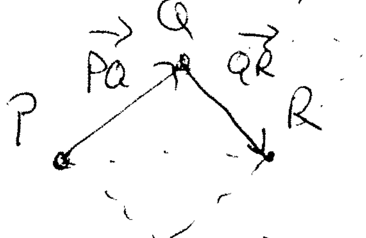
$h = \|\underline{u}\| \sin\theta$.

base $\times$ height $= \|\underline{u}\| \cdot \|\underline{v}\| \sin\theta = \|\underline{u} \times \underline{v}\|$ (By Th$^{\underline{m}}$)

Th$^{\underline{m}}$ The area of the $\parallel$gram maped out by two vectors $\underline{u}$ & $\underline{v}$ in $\mathbb{R}^3$ is $\|\underline{u} \times \underline{v}\|$

eg Find the area of the $\parallel$gram with consecutive pts

$P = (-1, 1, 1)$, $Q = (1, 0, 1)$ $R = (2, 2, 0)$



$\overrightarrow{PQ} = (2, -1, 0)$

$\overrightarrow{QR} = (1, 2, -1)$

$$\|\overrightarrow{PQ} \times \overrightarrow{QR}\| = \left\| \begin{vmatrix} i & j & k \\ 2 & -1 & 0 \\ 1 & 2 & -1 \end{vmatrix} \right\| = \|\underline{i}(1) - \underline{j}(-2) + k\,\overset{(4+1)}{5}\|$$

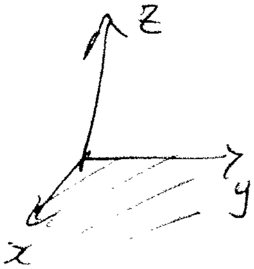
$$= \|(1, +2, 5)\|$$

$$= \sqrt{1^2 + 2^2 + 5^2} = \sqrt{1 + 4 + 25} = \underline{\underline{\sqrt{30}}}$$

Recall we had a th^{m} about the area of the ∥gram mapped out by 2 vectors in $\mathbb{R}^2$

$\underline{u} = (u_1, u_2)$ $\underline{v} = (v_1, v_2)$ area $= |u_1 v_2 - v_1 u_2|$

Proof: Embed $\mathbb{R}^2$ into the xy-plane of $\mathbb{R}^3$



- Add a 0 third component to each vector in $\mathbb{R}^2$

So $\underline{u}$ becomes $(u_1, u_2, 0)$ & $\underline{v}$ becomes $(v_1, v_2, 0)$

Know (now) how to do area in $\mathbb{R}^3$.

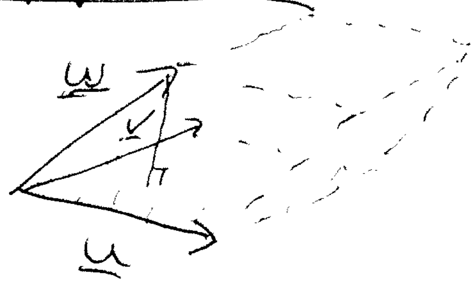
$$\|\underline{u} \times \underline{v}\| = \left\| \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} \right\| = \|0\underline{i} - 0\underline{j} + (u_2 v_1 - u_1 v_2)\underline{k}\|$$

$$= |u_1 v_2 - v_1 u_2|$$

(Note $\sqrt{(u_2 v_1 - u_1 v_2)^2} = |u_2 v_1 - u_1 v_2| = |u_1 v_2 - u_2 v_1|$)

Volume of a Parallelopiped in $\mathbb{R}^3$

Any 3 vectors in $\mathbb{R}^3$ map out a ∥piped



What is the volume?

= Area of base × height. Area of base $= \|\underline{u} \times \underline{v}\|$

height = component of $\underline{w}$ in a direct^{n} ⊥ to $\underline{u}$ & $\underline{v}$

$$= \mathrm{Comp}_{\underline{u}\times\underline{v}}\, \underline{w} = \frac{\underline{w}\cdot(\underline{u}\times\underline{v})}{\|\underline{u}\times\underline{v}\|}$$

$$\mathrm{Vol} = \|\underline{u}\times\underline{v}\| \frac{\underline{w}\cdot(\underline{u}\times\underline{v})}{\|\underline{u}\times\underline{v}\|} = |\underline{w}\cdot(\underline{u}\times\underline{v})|$$

5

Doesn't matter which face I choose as the 'base' get the same vol up to sign

$\underline{u} \cdot (\underline{v} \times \underline{w}) = (\underline{u} \times \underline{v}) \cdot \underline{w} = -(\underline{v} \times \underline{u}) \cdot \underline{w} = -\underline{v} \cdot (\underline{u} \times \underline{w})$

eg Find the Vol of the //piped mapped out by

$\underline{u} = (1,1,1) \quad \underline{v} = (0,1,0) \quad \underline{w} = (2,2,0)$

$$\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\underline{i} \cancel{+\underline{j}} + \underline{k}$$

$$(\underline{u} \times \underline{v}) \cdot \underline{w} = (-1, 0, 1) \cdot (2, 2, 0) = -2.$$

Volume $= \underline{2}$