Oct 26 09 (1)

<u>Th$^{\underline{m}}$</u> If $\underline{v}_1, \ldots, \underline{v}_n \in \mathbb{R}^n$ are lin indep.
Then If $A = (\underline{v}_1 \mid \underline{v}_2 \mid \ldots \mid \underline{v}_n)$ ← col's of A are $\underline{v}_1, \ldots, \underline{v}_n$
Then The system of eq$^{\underline{n}}$'s $A\underline{x} = \underline{0}$ has unique sol$^{\underline{n}}$
& $A\underline{x} = \underline{b}$ has unique sol$^{\underline{n}}$ for every $\underline{b} \in \mathbb{R}^n$

<u>Note</u> A is $n \times n$ i.e. square

Given $\underline{v}_1, \ldots, \underline{v}_m \in \mathbb{R}^n$ if $m > n$ $\underline{v}_1, \ldots, \underline{v}_m$ are lin dep.

eg 1) $\underline{v}_1 = (1,1,3)$ $\underline{v}_2 = (1,3,1)$ $\underline{v}_3 = (3,1,1)$ $\underline{v}_4 = (3,3,3)$
$\underline{v}_1, \ldots, \underline{v}_4$ are lin dep. since $4 > 3$

Basis & Dimension

<u>Def$^{\underline{n}}$</u> A set of vectors $\underline{v}_1, \ldots, \underline{v}_m \in \mathbb{R}^n$ is a basis for a subspace W iff

1) $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are lin. indep.
2) $\{\underline{v}_1, \ldots, \underline{v}_n\}$ ~~are a~~ span W.

2. Says any vector in W can be expressed as a lin. comb. of $\{\underline{v}_1, \ldots, \underline{v}_m\}$.
1. says we need all $\{\underline{v}_1, \ldots, \underline{v}_m\}$

Note we must have $m \leq n$

Think of W as being like a plane in $\mathbb{R}^3$

eg 1) Standard basis for $\mathbb{R}^n$. ②

$$\underline{e}_1 = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix},\ \underline{e}_2 = \begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix},\ \dots,\ \underline{e}_n = \begin{pmatrix}0\\0\\ \vdots\\0\\1\end{pmatrix}$$

$\underline{e}_i$ has a 1 in the i^{th} posit^n.

In fact any lin. indep. set of n vectors in $\mathbb{R}^n$ forms a basis.

eg 2) $\underline{v}_1 = (1,1,3)$ $\underline{v}_2 = (1,3,1)$ $\underline{v}_3 = (1,0,0)$
lin. indep. So span $\mathbb{R}^3$

3) $\underline{v}_1 = (1,1,3)$ $\underline{v}_2 = (1,3,1)$ $\underline{v}_3 = (2,4,2)$
$\underline{v}_1 + \underline{v}_2 - \underline{v}_3 = \underline{0}$ so lin. dep.
- Do not span $\mathbb{R}^3$
- Do span a plane with parametric form

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = s\begin{pmatrix}1\\1\\3\end{pmatrix} + t\begin{pmatrix}1\\3\\1\end{pmatrix}$$

4) $\underline{v}_1 = (1,1,3)$ $\underline{v}_2 = (1,3,1)$
Not a basis for $\mathbb{R}^3$ - not enough vectors to span.

Given a basis. $V = \{\underline{v}_1, \dots, \underline{v}_n\}$ in $\mathbb{R}^n$ we can write any vector $\underline{u} \in \mathbb{R}^n$ as a lin. combinat^n of $\underline{v}_1, \dots, \underline{v}_n$. i.e. There are scalars $a_1, \dots, a_n \in \mathbb{R}$ s.t. $\underline{u} = a_1\underline{v}_1 + \dots + a_n\underline{v}_n$.

The a_i are called the components of $\underline{u}$ w.r.t the basis V

eg Find the components of $\underline{u} = (2,4,4)$ w.r.t. the basis $V = \{ \underbrace{(1,1,3)}_{\underline{v}_1}, \underbrace{(1,3,1)}_{\underline{v}_2}, \underbrace{(1,0,0)}_{\underline{v}_3} \}$

Need to find a_1, a_2, a_3 s.t. $a_1\underline{v}_1 + a_2\underline{v}_2 + a_3\underline{v}_3 = \underline{u}$

$$a_1 \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + a_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 4 \end{pmatrix}$$

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 2 \\ a_1 + 3a_2 &= 4 \\ 3a_1 + a_2 &= 4 \end{aligned}\right\} \text{ -- 3 eq}^{\text{n}}\text{s in 3 vars}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 3 & 0 & 4 \\ 3 & 1 & 0 & 4 \end{array}\right)$$

$$(\underline{v}_1 \mid \underline{v}_2 \mid \underline{v}_3 \mid \underline{u})$$

Solve to get $a_1 = a_2 = 1$, $a_3 = 0$

So $\underline{u} = \underline{v}_1 + \underline{v}_2$

So $\underline{u} = (1,1,0)_V$ w.r.t. the basis V.

Th$^{\text{m}}$ If $\{\underline{u}_1, \dots, \underline{u}_{m_1}\}$ & $\{\underline{v}_1, \dots, \underline{v}_{m_2}\}$ are bases for a subspace W of $\mathbb{R}^n$ then $m_1 = m_2$

& This # is called the <u>dimension</u> of W.

Subspaces

contained in (4)

Def^n Given a subset of vectors $U \subseteq \mathbb{R}^n$

U is called a subspace if it is closed under vector addit^n & scalar mult^n.

i.e. 1) If $\underline{u}$ & $\underline{v} \in U$ then $\underline{u}+\underline{v} \in U$

2) $a \in \mathbb{R}$ $\underline{u} \in U$ then $a\underline{u} \in U$

Th^m $U \subseteq \mathbb{R}^n$ is a subspace iff. for every pair of vect $\underline{u}_1, \underline{u}_2 \in U$ & scalars $a_1, a_2 \in \mathbb{R}$

$$a_1\underline{u}_1 + a_2\underline{u}_2 \in U.$$

eg 1) $U = \{ \underbrace{\underline{u} \in \mathbb{R}^3}_{\text{big set}} \mid \underbrace{\underline{u} = t(1,1,2),\ t \in \mathbb{R}}_{\text{condit}^n \text{ for inclusion.}} \}$

the set, such that

[Must show closure]

Let $\underline{u}, \underline{v} \in U$.

$\Rightarrow$ $\underline{u} = t(1,1,2)$ & $\underline{v} = s(1,1,2)$ for some $s, t \in \mathbb{R}$

[Note Don't use t for both - o/w they would be equal]

Consider $\underline{u}+\underline{v} = t(1,1,2) + s(1,1,2)$ (Dist)

$= (s+t)(1,1,2)$

$s+t \in \mathbb{R}$. (form $k(1,1,2)$)

So $\underline{u}+\underline{v} \in U$.

Let $a \in \mathbb{R}$ Consider $a\underline{u} = at(1,1,2)$

$at \in \mathbb{R}$ so $a\underline{u} \in U$.

2) Let $\underline{u} = (1,0,1)$ & $\underline{v} = (2,1,1)$

$U = \{ \underline{w} \in \mathbb{R}^3 \mid \underline{w} = t\underline{u} + s\underline{v} \}$

Show U is a subspace.

Let $\underline{w}_1$ & $\underline{w}_2 \in U$

$\Rightarrow \underline{w}_1 = a_1\underline{u} + b_1\underline{v} \quad \underline{w}_2 = a_2\underline{u} + b_2\underline{v}$

$a_1, a_2, b_1, b_2 \in \mathbb{R}$

Consider $\underline{w}_1 + \underline{w}_2 =$

$= a_1\underline{u} + b_1\underline{v} + a_2\underline{u} + b_2\underline{v}$

$= (a_1 + a_2)\underline{u} + (b_1 + b_2)\underline{v}$

$a_1 + a_2 \in \mathbb{R}$ & $b_1 + b_2 \in \mathbb{R}$ — form $t\underline{u} + s\underline{v}$

So $\underline{w}_1 + \underline{w}_2 \in U$

Conside $a\underline{u}$ where $a \in \mathbb{R}$

$= a(a_1\underline{u} + b_1\underline{v})$

$= \underbrace{a a_1}_{\in\mathbb{R}}\underline{u} + \underbrace{ab_1}_{\in\mathbb{R}}\underline{v} \in U.$

<u>Note</u> $\underline{u}$ $(1,0,1)$ & $\underline{v} = (2,1,1)$ form a basis for the subspace U.

Any subspace has a basis. The size of that basis is called the dimension of the subspace.

- All lines through O are subspaces
- " plane " " " "

3) $U = \mathbb{R}^n$ in $\mathbb{R}^n$.

$\underline{u}, \underline{v} \in \mathbb{R}^n$ $\quad \underline{u}+\underline{v} \in \mathbb{R}^n$ & $a\underline{u} \in \mathbb{R}^n$ $(a \in \mathbb{R})$

So $\mathbb{R}^n$ is a subspace of itself.

4) $U = \{\underline{0}\} \subseteq \mathbb{R}^n$

$\underline{0} + \underline{0} = \underline{0} \in U \qquad a\underline{0} = \underline{0} \in U$

So $\{\underline{0}\}$ is a subspace. Called the trivial subspace

Def^n A subspace U of $\mathbb{R}^n$ is called a proper subspace if it is neither $\mathbb{R}^n$ or the trivial subspace.

$\underline{x} = (x, y, z)$

5) $U = \{\underline{x} \in \mathbb{R}^3 \mid ax + by + cz = 0,\ a, b, c, d \in \mathbb{R}\}$

Show U is a subspace.

Let $\underline{u}, \underline{v} \in U$ $\qquad au_1 + bu_2 + cu_3 = 0 \qquad \underline{u} \cdot \underline{n} = 0$

$\qquad\qquad\qquad\qquad av_1 + bv_2 + cv_3 = 0 \qquad \underline{v} \cdot \underline{n} = 0$

Consider $(\underline{u} + \underline{v}) \cdot \underline{n} = \underline{u} \cdot \underline{n} + \underline{v} \cdot \underline{n}$

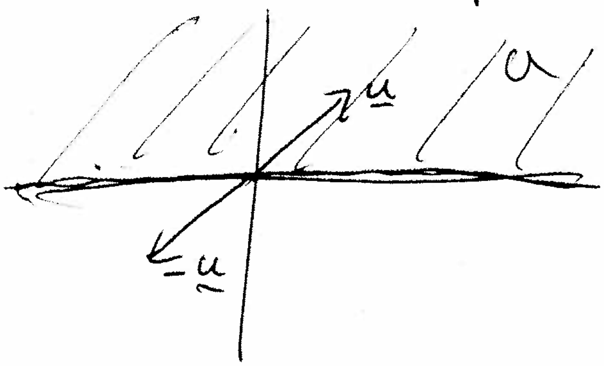
$= 0 + 0 = 0$

So $\underline{u} + \underline{v} \in U$.

Sim. $a\underline{u} \cdot \underline{n} = 0$ so $a\underline{u} \in U$

All planes through 0 in $\mathbb{R}$ are subspaces.

6) (Not a subspace) $U = \{ \underline{u} \in \mathbb{R} \mid u_1 > 0 \}$ $\underline{u} = (u_1, u_2)$



Upper Half plane

Consider $0\underline{u} = \underline{0} \notin U$ $(0,0)$

So $0\underline{u} \notin U$

Th^{m} If $\underline{0} \notin U$ then U is <u>not</u> a subspace,

7) $U = \{ u \in \mathbb{R} \mid u_1 \geq 0 \}$

Now $\underline{0} \in U$ But U is not a subspace

Let $\underline{u} \in U$ with $u_1 > 0$

Consider $(-1)\,\underline{u} = (-u_1, -u_2)$ $u_1 \neq 0$

$u_1 > 0 \Rightarrow -u_1 < 0$

So $(-1)\underline{u} \notin U$

Consider $\overline{(-1)(1,2)} = (-1,-2) \notin U$

a $\quad \underline{u} \in U$

Thm Given a homogeneous system $A\underline{x} = \underline{0}$ with n unknowns (A has n cols) then its solut^{n} set is a subspace of $\mathbb{R}^n$.

eg

$$\begin{aligned} x_1 + x_3 + x_4 &= 0 \\ x_1 + x_2 + x_4 &= 0 \\ 2x_1 + x_2 + x_3 + 2x_4 &= 0 \\ 3x_1 + 2x_2 + x_3 + 3x_4 &= 0 \end{aligned} \qquad A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & 2 & 1 & 3 \end{pmatrix}$$

Solve

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & 3 & 0 \end{array}\right) \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 2 & -2 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $s, t \in \mathbb{R}$ $\quad x_3 = s$, $x_4 = t$

$x_2 = s \quad x_1 = -s - t$.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \underbrace{\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}}_{v_1} + t \underbrace{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{v_2}$$

i.e. Sol^{n} ~~set~~ is $\mathrm{sp}\{\underline{v}_1, \underline{v}_2\}$

CAA