Showing a Transformat^n is linear 16-Nov 09

eg 1) $T(x,y,z) = (kx, ky, kz)$ — Dilat^n by k.

Show T is Linear

[Note $T(0,0,0) = (k0, k0, k0) = (0,0,0) = \underline{0}$]

Let $\underline{u} = (u_1, u_2, u_3)$ & $\underline{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ & Let $c, d \in \mathbb{R}$

Consider $T(c\underline{u} + d\underline{v}) = T(c(u_1, u_2, u_3) + d(v_1, v_2, v_3))$

$= T(cu_1 + dv_1, cu_2 + dv_2, cu_3 + dv_3)$

$= (k(cu_1 + dv_1), k(cu_2 + dv_2), k(cu_3 + dv_3))$

$= kc(u_1, u_2, u_3) + kd(v_1, v_2, v_3)$

$= cT(\underline{u}) + dT(\underline{v})$

[$cT(\underline{u}) + dT(\underline{v}) = ck\underline{u} + dk\underline{v}$]

So $T(c\underline{u} + d\underline{v}) = cT(\underline{u}) + dT(\underline{v})$.

So linear. $\square$

Method 2.

Let $\underline{u} \in \mathbb{R}^3$ & $\underline{v} \in \mathbb{R}^3$ $c \in \mathbb{R}$

1) Consider $T(\underline{u} + \underline{v}) = T(u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$= k(u_1 + v_1, u_2 + v_2; u_3 + v_3)$

$= k(\underline{u}) + k\underline{v}$

$= T(\underline{u}) + T(\underline{v})$

So $T(\underline{u} + \underline{v}) = T(\underline{u}) + T(\underline{v})$

2) Consider $T(c\underline{u}) = T(cu_1, cu_2, cu_3)$

$= (cku_1, cku_2, cku_3)$

$= c(k\underline{u})$

$= cT(\underline{u})$

So $T(c\underline{u}) = cT(\underline{u})$.

Thus Properties 1 & 2 hold, so linear.

Finding the Matrix associated with a given Linear Transformat^

eg 1) $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ $\quad A\underline{x} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_3 + x_4 \end{pmatrix}$

So $T_A(\underline{x}) = (x_1 + x_2, x_3 + x_4)$

2). Given $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad T(x,y) = (x, y, x+y)$

Find A s.t. $A = [T]$

$T(\underline{e}_1) = T(1,0) = (1, 0, 1)$

$T(\underline{e}_2) = T(0,1) = (0, 1, 1)$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

<u>Check</u> $\quad A\underline{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$

3) Given That $T(\underline{e}_1) = (1,1,1)$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^3$

$T(\underline{e}_2) = (0,1,0)$

$T(\underline{e}_3) = (1,1,0)$

a) Find A s.t. $T = T_A$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

b) Find $T(1,1,0)$

$$T(1,1,0) = A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underline{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}$$

c) Find $T(x,y,z)$

$$T(x,y,z) = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ x+y+z \\ x \end{pmatrix}$$

4) Let $\underline{u} = (1,2)$ $\underline{v} = (-1,1)$
Define $T(x,y) = x\underline{u} + y\underline{v}$

a) Find $T(1,-1)$ –

$$T(1,-1) = \underline{u} + (-1)\underline{v} = \underline{u} - \underline{v} = (1,2) - (-1,1)$$
$$= (2,1)$$

b) Show T is linear.

Consider $T(c(x,y) + d(x',y'))$ $\quad (x, y, x', y', c, d \in \mathbb{R})$

$$= T(cx + dx', cy + dy')$$
$$= (cx + dx')(\underset{\underline{u}}{1,2}) + (cy + dy')(\underset{\underline{v}}{-1,1}) \quad \left(\begin{array}{c}\text{Redistribute} \\ c \text{ part } \& \ d \text{ part}\end{array}\right)$$
$$\cancel{= (cx + cy)}$$
$$\cancel{= cx\underline{u} + dy\underline{v}}$$
$$cx\underline{u} + cy\underline{v} + dx'\underline{u} + dy'\underline{v} \quad (\text{Dist})$$
$$= c(x\underline{u} + y\underline{v}) + d(x'\underline{u} + y'\underline{v}) \quad (\text{Def}^{n} \text{ of } T)$$
$$= c\,T(x,y) + d\,T(x',y')$$

c) Find A s.t. $A = [T]$ – The matrix associated with T

$$T(\underline{e}_1) = T(1,0) = \underline{u} = (1,2)$$
$$T(\underline{e}_2) = T(0,1) = \underline{v} = (-1,1)$$
$$\text{Th } A = (\underline{u} \mid \underline{v}) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$$

<u>Check</u> $\quad A\underline{x} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ 2x + y \end{pmatrix} = x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$= x\underline{u} + y\underline{v}.$$

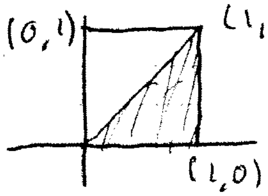
eg $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Geometry of Linear Operators in $\mathbb{R}^2$

Consider a $T : \mathbb{R}^2 \to \mathbb{R}^2$ T linear

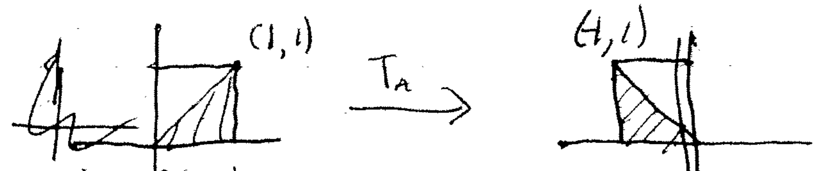
So $[T] = A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Want to consider what happens to the unit square for various values of $a, b, c, d \in \mathbb{R}$.



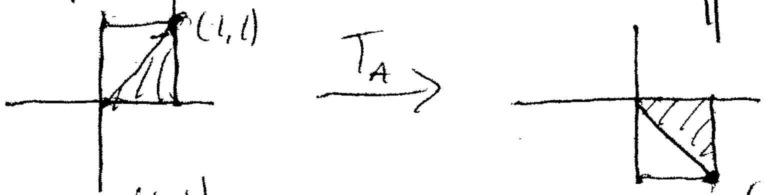
— Unit Square.

eg 1) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



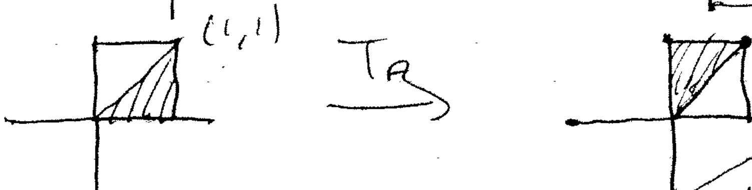
Reflect^n about y-axis.

2) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



Reflect^n about x-axis

3) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



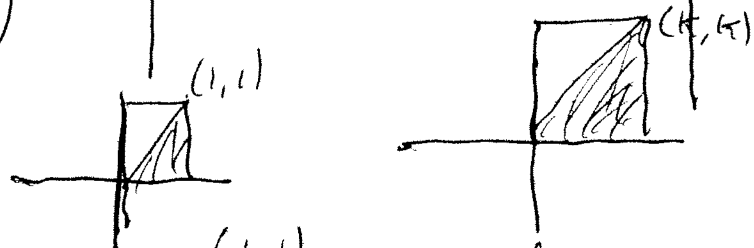
Reflect^n about $y = x$.

$T(x, y) = (y, x)$

4) $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

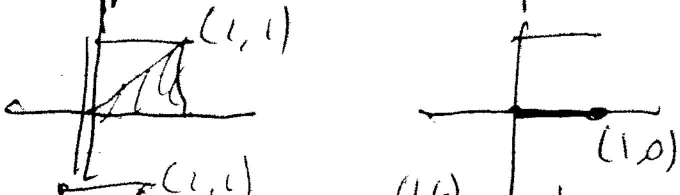
Rotat^n by θ.

5) $\begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$



Dilat^n by K.

6) $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

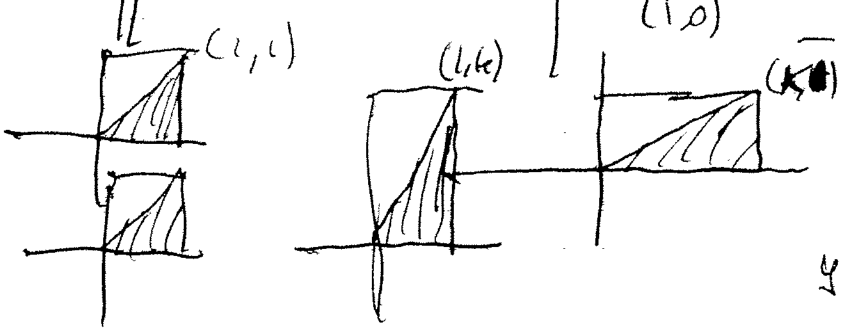


Project^n onto x-axis.

— Noninvertable

7) $\begin{pmatrix} K & 0 \\ 0 & 1 \end{pmatrix}$

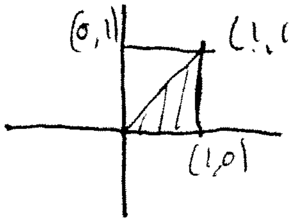
8) $\begin{pmatrix} 1 & 0 \\ 0 & K \end{pmatrix}$



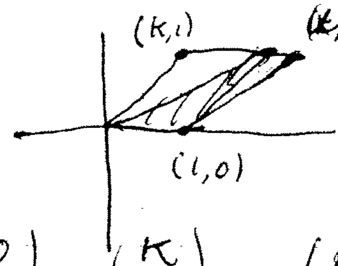
x-expansion by factor K

y-expansion factor K

(5) 9) $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$



$\xrightarrow{T_A}$

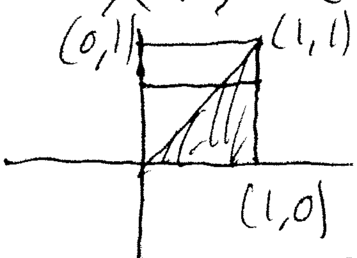


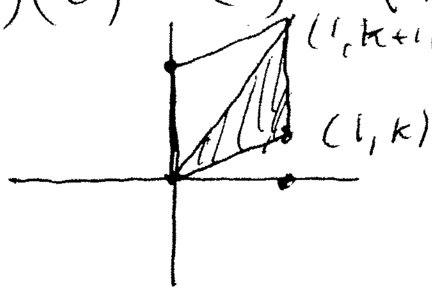
$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+k \\ \cancel{\ } \end{pmatrix}$

x-shear with factor k

10) $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ k \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ k+1 \end{pmatrix}$





y-shear factor k

<u>Th^{m}</u> If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a linear invertable trans. then.

a) T maps straight lines to straight lines.
b) T " lines through 0 " lines through 0
c) T " parallel lines to parallel lines
d) T " the line segment PQ to the line segment T(P)T(Q)
e) 3 pts lie on a line iff their images all lie on a line