Nov 11 09 (i)

Eigen Values & Eigen Vectors

eg

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Find the eigenvalues & e.vectors of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda-1 & 1 & 0 \\ 1 & \lambda-2 & 1 \\ 0 & 1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-2)(\lambda-1)-1\right] - (\lambda-1)$$

$$= (\lambda-1)\left[(\lambda-2)(\lambda-1)-1-1\right]$$

$$= (\lambda-1)(\lambda^2-3\lambda+2-2)$$

$$= (\lambda-1)(\lambda^2-3\lambda)$$

$$= \lambda(\lambda-1)(\lambda-3)$$

So e.values are $\lambda=0$, $\lambda=1$ & $\lambda=3$

$\underline{\lambda=0}$ Solve $(\lambda I-A)\underline{x}=\underline{0} \Rightarrow -A\underline{x}=\underline{0}$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & +1 & -1 & 0 \end{array}\right) \quad R_2 \to R_2+R_1$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \quad R_3 \to R_3+R_2$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$ $z=t$, $y=t$, $x=t$

$E_0 = \{ t(1,1,1) \mid t \in \mathbb{R} \}$

$\lambda = 1$ Solve $(I - A)\underline{x} = \underline{0}$

2

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad R_2 \leftrightarrow R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \quad R_3 \to R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$, $z = t$, $y = 0$, $x = -t$

$E_1 = \{ t(-1, 0, 1) \mid t \in \mathbb{R} \}$

$\lambda = 3$

$$\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \quad R_2 \leftrightarrow R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \quad R_2 \to R_2 - 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right) \quad R_3 \to R_3 + R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $t \in \mathbb{R}$ $z = t$, $y = -2t$

$x = -(-2t) - t = 2t - t$

$x = t$

$E_3 = \{ t(1, -2, 1) \mid t \in \mathbb{R} \}$

3 e. values $\lambda = 0$, $\lambda = 1$ & $\lambda = 3$

$\underset{\lambda=0}{(1,1,1)}$, $\underset{\lambda=1}{(-1,0,1)}$, $\underset{\lambda=3}{(1,-2,1)}$ form a basis for $\mathbb{R}^3$

i.e. Any vector in $\mathbb{R}^3$ can be expressed as a lin comb. of these

General Transformat$^{\underline{n}}$s

(3)

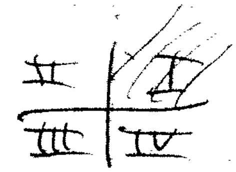
eg 1) $T(x,y) = (x+1, y)$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$

$\mathrm{Dom}(T) = \mathbb{R}^2$

$\mathrm{Ran}(T) = \mathbb{R}^2$

$\mathrm{Ker}(T) = \{(-1,0)\}$

2) $T(x,y) = (|x|, |y|)$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^2$



$\mathrm{dom}(T) = \mathbb{R}^2$

$\mathrm{ran}(T) = \{(x,y) \in \mathbb{R} \mid x \geq 0 \ \& \ y \geq 0\}$ – (I'st quadrant)

$\mathrm{Ker}(T) = \{(0,0)\}$

3) $T(x,y) = (x, y, x+y)$ $\quad T: \mathbb{R}^2 \to \mathbb{R}^3$

$\mathrm{Dom}(T) = \mathbb{R}^2$

$\mathrm{Ran}(T) = \{(x,y,z) \mid z = x+y\}$ – Plane $x+y-z=0$

$\mathrm{Ker}(T) = \{(x,y) \in \mathbb{R}^2 \mid T(x,y) = 0\}$

Note that $T(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}$

So $\mathrm{Ker}(T)$ is the sol$^{\underline{n}}$ to the homogeneous system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{Ker}(T) = \{\underline{0}\} = \{(0,0)\}$$

4) $T(x,y,z) = (x+z, y+z)$ $\quad T: \mathbb{R}^3 \to \mathbb{R}^2$

$\mathrm{Dom}(T) = \mathbb{R}^3$

$\mathrm{Ran}(T) = \mathbb{R}^2$

$\mathrm{Ker}(T) = \{t(-1,-1,1) \mid t \in \mathbb{R}\}$

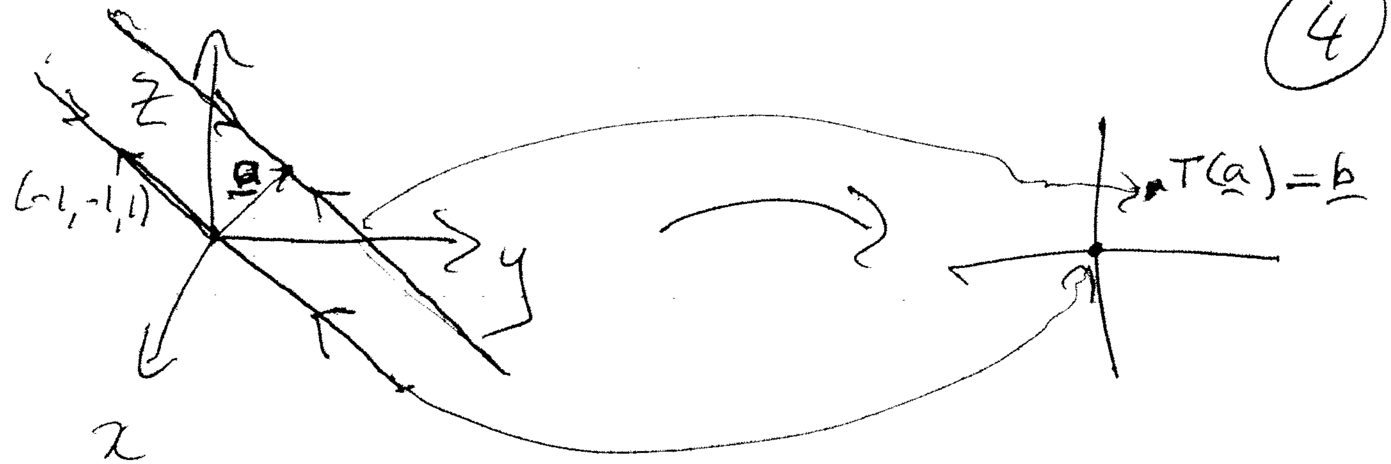
= line // to $(-1,-1,1)$ through O

$T(x,y,z) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+z \\ y+z \end{pmatrix}$

Solve Homogeneous system

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Get $t \in \mathbb{R}$, $z = t$, $y = -t$, $x = -t$



<u>Recall</u> if $\underline{h}$ is a sol^n to $A\underline{x} = \underline{0}$ ~~then~~ & ~~that~~ $A\underline{a} = \underline{b}$ then $A(\underline{a}+\underline{h}) = \underline{b}$

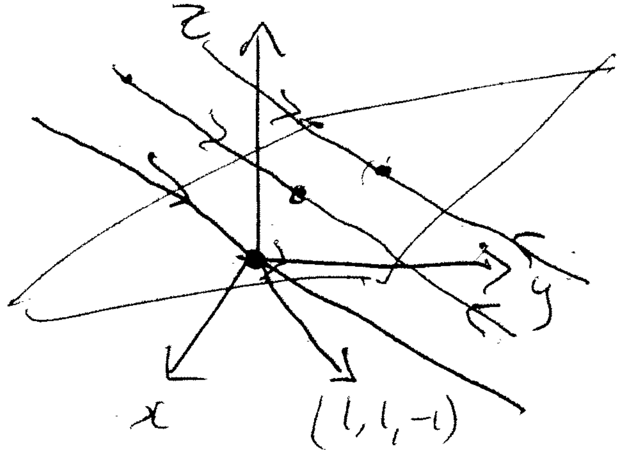
$\Leftrightarrow \quad T(\underline{h}) = \underline{0}$ & $T(\underline{a}) = \underline{b} \Rightarrow T(\underline{a}+\underline{h}) = \underline{b}$

Last 2 eg's had special property that they were '<u>Matrix Transformat^n's</u>'

<u>eg</u> 2 From Notes p.4 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

$\mathrm{Ran}(T_A) = \{ \underline{x} \in \mathbb{R}^3 \mid x + y - z = 0 \}$

$\mathrm{Ker}(T_A) = \{ x \in \mathbb{R}^3 \mid t(1,-1,1) = \underline{x} \}$



Matrix Transformations

eg 1) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}$ $\quad T_A : \mathbb{R}^2 \to \mathbb{R}^3$ (A is 3×2)

$$T_A(\underline{x}) = A\underline{x} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ 2x+y \end{pmatrix}$$

So $T_A(x, y) = (x+y, y, 2x+y)$

a) Find $\ker(T_A)$ $\quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{array}\right)$ $R_3 \to R_3 - 2R_1$

$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right)$ $R_3 \to R_3 + R_2$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ Soln $\{\underline{0}\}$

$\ker(T_A) = \{(0,0)\}$

b) Find $\mathrm{Ran}(T_A)$ $\quad \left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & 1 & y_2 \\ 2 & 1 & y_3 \end{array}\right)$ $R_3 \to R_3 - 2R_1$

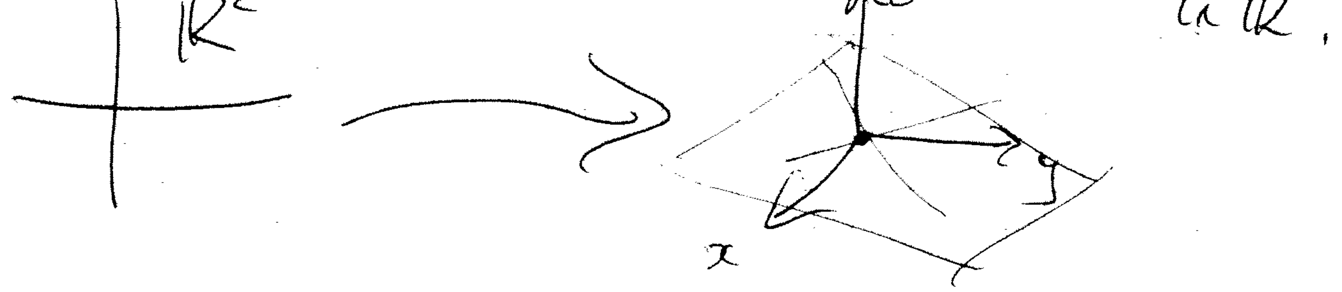
$\left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & 1 & y_2 \\ 0 & -1 & y_3 - 2y_1 \end{array}\right)$ $R_3 \to R_3 + R_2$ $\left(\begin{array}{cc|c} 1 & 1 & y_1 \\ 0 & 1 & y_2 \\ 0 & 0 & y_3 - 2y_1 + y_2 \end{array}\right)$ $= 0$

$\mathrm{Ran}(T_A) = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y - z = 0\}$

Range is plane $2x - y - z = 0$

c) Describe the map T_A

T_A maps $\mathbb{R}^2$ into the plane $2x - y - z = 0$ in $\mathbb{R}^3$.



6

2) $A=\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}$ $T_A: \mathbb{R}^3 \to \mathbb{R}^2$ (A is 2x3)

a) Find $T_A(1,1,1)$

$$\begin{pmatrix}1&1&0\\0&1&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\2\end{pmatrix}$$ So $T_A(1,1,1)=(2,2)$

b) Find $\ker(~~A~~T_A)$ (Solve hom. sys. $A\underline{x}=\underline{0}$)

$$\begin{pmatrix}1&1&0&|&0\\0&1&1&|&0\end{pmatrix}$$

Let $t\in\mathbb{R}$ $z=t$, $y=-t$, $x=t$

$\ker(T_A)=\{\underline{x}\in\mathbb{R}^3 \mid x=t(1,-1,1), t\in\mathbb{R}\}$

- Line // to $(1,-1,1)$ through O.

c) Find $\mathrm{Ran}(T_A)$.

$$\begin{pmatrix}1&1&0&|&y_1\\0&1&1&|&y_2\end{pmatrix}$$ — Sol^n for every $(y_1,y_2)\in\mathbb{R}^2$

3) Given $A=\begin{pmatrix}1&1\\0&1\\1&2\end{pmatrix}$ a) is $\underline{b}=(1,1,1)\in\mathrm{Ran}(T_A)$?

~~Solu~~ Find if there is a sol^n for $A\underline{x}=\underline{b}$

$$\begin{pmatrix}1&1&|&1\\0&1&|&1\\1&2&|&1\end{pmatrix} R_3\to R_3-R_1 \begin{pmatrix}1&1&|&1\\0&1&|&1\\0&1&|&0\end{pmatrix} R_3\to R_3-R_2$$

$$\begin{pmatrix}1&1&|&1\\0&1&|&1\\0&0&|&-1\end{pmatrix}$$ No Sol^n $(0=-1)$

So $(1,1,1)\notin\mathrm{Ran}(T_A)$

b) Is $(2,-1,1)\in\mathrm{Ran}(T_A)$

$$\begin{pmatrix}1&1&|&2\\0&1&|&-1\\1&2&|&1\end{pmatrix} R_3\to R_3-R_1 \begin{pmatrix}1&1&|&2\\0&1&|&-1\\0&1&|&-1\end{pmatrix} R_3\to R_3-R_2 \begin{pmatrix}1&1&|&2\\0&1&|&-1\\0&0&|&0\end{pmatrix}$$

So Sol^n exists & $(2,-1,1)\in\mathrm{Ran}(T_A)$