Vectors and Matrices

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1 Vectors and Matrices

1.1 Definitions

Definition 1

1. A Matrix is an $m \times n$ (m by n) array of numbers.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

2. A Vector is a $1 \times n$ or $n \times 1$ matrix. That is an ordered set of n numbers.

   We say that such a vector is of dimension n.

3. A scalar is a number (usually either real or complex).

Notation 2

- We generally use uppercase letters from the beginning of the alphabet $(A, B, C \dots)$ to denote matrices.
- We generally use lowercase boldface letters from the end of the alphabet $(\mathbf{u}, \mathbf{v}, \mathbf{w} \dots)$ to denote vectors.
- We use the convention that $\mathbf{u} = (u_1, u_2, \dots, u_n)$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$, etc.
- If $\mathbf{x} = (x_1, x_2, \dots, x_n)$ then the scalars $x_1, x_2, \dots, x_n$ are called the components of $\mathbf{x}$.
- We denote the set of all vectors of dimension n whose components are real numbers by $\mathbb{R}^n$.
- We denote the set of all vectors of dimension n whose components are complex numbers by $\mathbb{C}^n$.

Note This definition of vector differs from the usual 'High School' definition involving magnitude and direction.

1.2 Special Matrices and Vectors

1. **The Identity matrix**
   The identity matrix is a square matrix with 1's down the diagonal, and zeros elsewhere. The $n \times n$ identity matrix is denoted I_n.

$$I_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

2. **The Zero Matrix**
   The zero matrix is an $m \times n$ matrix, all of whose entries are 0.

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

3. **The Zero Vector**
   The zero vector is a vector, all of whose entries are 0.

$$\mathbf{0} = (0, 0, \dots, 0)$$

4. **Elementary Vectors** An elementary vector, $\mathbf{e}_i$ is a vector which has zeros everywhere, except in the i^{th} position, where it is one.

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \dots, 0) \\ \mathbf{e}_2 &= (0, 1, \dots, 0) \\ &\vdots \\ \mathbf{e}_i &= (0, 0, \dots, 1, \dots, 0) \qquad 1 \text{ in } i^{\text{th}} \text{ position} \\ &\vdots \\ \mathbf{e}_n &= (0, 0, \dots, 1) \end{aligned}$$

2 Operations on Matrices

1. **Transpose**
   Given an $m \times n$ matrix, A, the transpose of A is obtained by interchanging the rows and columns of A. We denote the transpose of A by A^t, or A^T.

   Note that if A is $m \times n$ then A^t will be $n \times m$.

Example 3

(a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

(c)

$$(1, 2, 3)^t = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

2. **Matrix Addition**

Given two $m \times n$ matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

We may define the sum of A and B, $A + B$, to be the sum componentwise, i.e.

$$A + B = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

This works for vectors as well.

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

Note that matrix addition is only defined if A and B have the same size.

Example 4

(a)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} + \begin{pmatrix} 10 & 11 & 12 \\ 13 & 14 & 15 \\ 16 & 17 & 18 \end{pmatrix} = \begin{pmatrix} 1+10 & 2+11 & 3+12 \\ 4+13 & 5+14 & 6+15 \\ 7+16 & 8+17 & 9+18 \end{pmatrix} = \begin{pmatrix} 11 & 13 & 15 \\ 17 & 19 & 21 \\ 23 & 25 & 27 \end{pmatrix}$$

(b)

$$(1,2,3)+(4,5,6)=(1+4,2+5,3+6)=(5,7,9)$$

3. **Matrix Multiplication**

(a) **Scalar Multiplication**

Given a matrix A, and a scalar k, we define the scalar product of k with A, kA by multiplying each entry of A by k.

$$kA = k\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}$$

Note that this works for vectors as well.

$$k\mathbf{u} = k(u_1, u_2, \dots, u_n) = (ku_1, ku_2, \dots, ku_n)$$

Examples

i.

$$10\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{pmatrix}$$

ii.

$$5(1,2,3) = (5,10,15)$$

(b) **Vector Scalar Product or Dot Product**

Given two n dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ we define the vector scalar product or dot product of $\mathbf{u}$ and $\mathbf{v}$ as the sum of the product of the components. So

$$\mathbf{u}\cdot\mathbf{v} = (u_1, u_2, \dots, u_n)\cdot(v_1, v_2, \dots, v_n) = u_1v_1 + u_2v_2 + \dots + u_nv_n.$$

Note that the dot product is defined only for vectors, furthermore the dot product of two vectors yields a scalar.

Example 5

$$(1,2,3)\cdot(4,5,6) = 1\times 4 + 2\times 5 + 3\times 6 = 4 + 10 + 18 = 32$$

Definition 6

i. The dot product of a vector $\mathbf{u}$ with itself ($\mathbf{u}\cdot\mathbf{u}$) is the square of the length or magnitude of $\mathbf{u}$. We write $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$.

ii. A vector, $\mathbf{u}$, of length one is called a unit vector. Note that a unit vector satisfies $\mathbf{u} \cdot \mathbf{u} = 1$.

Example 7
Find the magnitude of the vector $\mathbf{u} = (1, 2, 3)$

$$\mathbf{u} \cdot \mathbf{u} = (1, 2, 3) \cdot (1, 2, 3) = 1 + 2 + 9 = 14$$

Thus $|\mathbf{u}| = \sqrt{14}$.

Theorem 8 *Given two n dimensional vectors $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.*

(c) **Matrix Multiplication**

If A and B are two matrices where A has the same number of columns as B has rows (i.e. A is $m \times n$ and B is $n \times r$) we define the matrix product, AB to be the matrix in which the i, j^{th} entry is made up of the dot product of the i^{th} row of A with the j^{th} column of B.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nr} \end{pmatrix}$$

$$AB = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1r} \\ b_{21} & b_{22} & \dots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nr} \end{pmatrix} =$$

$$\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1n}b_{n1} & \dots & a_{11}b_{1r} + a_{12}b_{2r} + \ldots + a_{1n}b_{nr} \\ a_{21}b_{11} + a_{22}b_{21} + \ldots + a_{2n}b_{n1} & \dots & a_{21}b_{1r} + a_{22}b_{2r} + \ldots + a_{2n}b_{nr} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \ldots + a_{mn}b_{n1} & \dots & a_{m1}b_{1r} + a_{m2}b_{2r} + \ldots + a_{mn}b_{nr} \end{pmatrix}$$

Example 9

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 9 + 2 \times 6 + 3 \times 3 & 1 \times 8 + 2 \times 5 + 3 \times 2 & 1 \times 7 + 2 \times 4 + 3 \times 1 \\ 4 \times 9 + 5 \times 6 + 6 \times 3 & 4 \times 8 + 5 \times 5 + 6 \times 2 & 4 \times 7 + 5 \times 4 + 6 \times 1 \\ 7 \times 9 + 8 \times 6 + 9 \times 3 & 7 \times 8 + 8 \times 5 + 9 \times 2 & 7 \times 7 + 8 \times 4 + 9 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 + 12 + 9 & 8 + 10 + 6 & 7 + 8 + 3 \\ 36 + 30 + 18 & 32 + 25 + 12 & 28 + 20 + 6 \\ 63 + 48 + 27 & 56 + 40 + 18 & 49 + 32 + 9 \end{pmatrix} = \begin{pmatrix} 30 & 24 & 18 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{pmatrix}$$

Note that Matrix multiplication is only defined if A has the same number of columns as B has rows.

Example 10

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem 11 (1.4.1)

i)	$(k+\ell)A = kA + \ell A$	(Distributivity of scalar multiplication I)
i)	$k(A+B) = kA + kB$	(Distributivity of scalar multiplication II)
i)	$A(B+C) = AB + AC$	(Distributivity of matrix multiplication)
i)	$A(BC) = (AB)C$	(Associativity of matrix multiplication)
i)	$A+B = B+A$	(Commutativity of matrix addition)
i)	$(A+B)+C = A+(B+C)$	(Associativity of matrix addition)
i)	$k(AB) = A(kB)$	(Commutativity of Scalar Multiplication)

BIG Note

Matrix multiplication is **NOT** commutative. i.e. It is **NOT** true that $AB = BA$ (where defined).