Gaussian Elimination

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1 Row Echelon Form

Definition 1 *1. A matrix is in Row Echelon Form (REF) if all of the following hold:*

(a) Any rows consisting entirely of 0's appear at the bottom.

(b) In any non-zero row the first number, from the left, is a one. Called the leading one *or* pivot.

(c) In any two successive non-zero rows the leading one on top is to the left of the one on the bottom.

2. A matrix is in Reduced Row Echelon Form (RREF) if it is in REF (all of the above hold) and any column containing a leading one is zero in all other entries.

2 The Gaussian Algorithm

The following Algorithm reduces an $n \times m$ matrix to REF by means of elementary row operations alone.

1. For Each row i (R_i) from 1 to n

   (a) If any row j below row i has non zero entries to the right of the first non zero entry in row i exchange row i and j ($R_i \leftrightarrow R_j$) [Ensure We are working on the leftmost nonzero entry.]

   (b) Preform $R_i \to \frac{1}{c} R_i$ where $c =$ the first non-zero entry of row i. [This ensures that row i starts with a one.]

   (c) For each row j (R_j) below row i (Each $j > i$)

      i. Preform $R_j \to R_j - dR_i$ where $d =$ the entry in row j which is directly below the pivot in row i. [This ensures that row j has a 0 below the pivot of row i.]

   (d) If any 0 rows have appeared exchange them to the bottom of the matrix.

3 The Gaussian-Jordan Algorithm

The following Algorithm reduces an $n \times m$ matrix to RREF by means of elementary row operations alone.

1. Preform Gaussian elimination to get the matrix in REF

2. For each non zero row i (R_i) from n to 1 (bottom to top)
   (a) For each row j (R_j) above row i (Each $j < i$)
      i. Preform $R_j \to R_j - bR_i$ where $b =$ the value in row j directly above the pivot in row i. [This ensures that row j has a zero above the pivot in row i]

Example of Gaussian Elimination and The Gauss-Jordan Method

Solve the following system of equations.

$$\begin{array}{rcl} x_1 + 3x_2 - 2x_3 + 2x_5 &=& 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &=& -1 \\ 5x_3 + 10x_4 + 15x_6 &=& 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &=& 6 \end{array}$$

The Augmented Matrix is:

$$\left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right)$$

First leading 1 is in the 1,1 position, already 1.
Get all 0's below this leading 1 position.

$$\begin{array}{l} R_2 \longrightarrow R_2 - 2R_1 \\ R_4 \longrightarrow R_4 - 2R_1 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Get leading 1 in second row.

$$R_2 \longrightarrow -R_2 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{array}\right)$$

Get all 0's below second leading 1.

$$\begin{array}{l} R_3 \longrightarrow R_3 - 5R_2 \\ R_4 \longrightarrow R_4 - 4R_2 \end{array} \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right)$$

Move row of 0's to bottom:

$$R_3 \leftrightarrow R_4 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Get next leading 1.

$$R_3 \longrightarrow \frac{1}{6}R_3 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Matrix is now in Row Echelon Form.

Gauss Elimination

We now use back substitution. The Matrix translates to the following system of equations:

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\ x_3 + 2x_4 + 3x_6 &= 1 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

For each variable corresponding to a column not containing a leading 1, we assign a free variable.
Let $s, t, r \in \mathbb{R}$.
Let $x_2 = s, x_4 = t, x_5 = r$.
Then the equations imply: $x_6 = \frac{1}{3}$
$x_3 = 1 - 2x_4 - 3x_6 = 1 - 2t - 1 = -2t$ So $x_3 = -2t$.
$x_1 = -3x_2 + 2x_3 - 2x_5 = -3s + 2(-2t) - 2r$. So $x_1 = -3s - 4t - 2r$.
Thus the final solution is:
$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, s, -2t, t, r, \frac{1}{3})$

Gauss-Jordan

We continue the algorithm to get the matrix in Reduced Row Echelon Form.
Get 0's above rightmost leading 1 (in column 6).

$$R_2 \longrightarrow R_2 - 3R_3 \left(\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Get 0's above next leading 1 (in column 3).

$$R_1 \longrightarrow R_1 + 2R_2 \left(\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The Matrix is now in Reduced Row Echelon Form.
The Matrix translates to the following system of equations:

$$\begin{aligned} x_1 + 3x_2 + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

For each variable corresponding to a column not containing a leading 1, we assign a free variable.
Let $s, t, r \in \mathbb{R}$.
Let $x_2 = s, x_4 = t, x_5 = r$.
Then the matrix implies: $x_6 = \frac{1}{3}$
$x_3 = -2t$
$x_1 = -3x_2 - 4x_4 - 2x_5 = -3s - 4t - 2r$.
Thus the final solution is
$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, s, -2t, t, r, \frac{1}{3})$.