“How long do you want to live?”
Cops and Robber on Infinite Graphs

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Happy Birthday Robert!
Sahara Desert near Tozeur, Tunisia, May 2008

Dar el Jeld Restaurant, Tunis
Game of Cops and Robber

\[ \text{R loses} \]
• some graphs need more than one cop

• cop (or search) number of $G$, $c(G)$ (Aigner, Fromme, 84)
  – girth($G$) $\geq 5$, then $c(G) \geq \delta(G)$
  – if $G$ is planar, then $c(G) \leq 3$

• structure of cop-win ($c(G) = 1$) graphs is well-known and beautiful
  (Nowakowski, Winkler, 83), (Quilliot, 78)
• **cops and robber** is a simplified model for network security:
  – cops *(authorities/network administrator/antivirus software)* guard network from the robber *(intruder/hacker/virus)*
  – the larger the cop number, the less secure the network

• large literature on the game and its variants:

  *surveys*: Alspach, 2006; Hahn, 2007; Fomin, 2008
How big can $c(G)$ be?

- **Meyniel Conjecture:** For a connected graph $G$ of order $n$,
  
  $c(G) = O(n^{1/2})$.

- Graphs $G$ with $c(G) = \Theta(n^{1/2})$ may be constructed from projective planes.

- **(Chiniforooshan, 2008):** $c(G) \leq O(n / \log n)$
  
  - Upper bound of $n^{1-c}$ is open, $c > 0$

- **(Bonato, Hahn, Wang, 07):** Graph of order $n$ chosen uniformly at random has cop number $\Theta(\log n)$
Complexity

- (Berrarducci, Intrigila, 93), (Goldstein, Reingold, 95), (Hahn, MacGillivray, 03)

1. “c(G) ≤ k?” k fixed: in time $O(n^{O(k)})$

- (Fomin, Golovach, Kratochvíl, Nisse, Suchan, 08): computing the cop number of a given graph is:

2. NP-hard; W[2]-hard
3. subexponential time: $2^{o(n)}n^{O(1)}$
Structure of $k$-cop-win graphs

- find a structural characterization akin to dismantling for graphs with cop number $> 1$
  - even cop number 2 is open
- question: there is a polytime algorithm to recognize $c(G) \leq k$ (k fixed), but what is the structure theorem?
  - Winkler vs Nowakowski (!)
Cops and Robber on infinite graphs:
How long do you want to live?
Characterization in the infinite case

- **folklore**: for bipartite graphs $G$:
  
  $G$ is cop-win iff $G$ is a rayless tree

- dismantling characterization, however, does not apply in the infinite case
Nowakowski, Winkler Characterization

- for each vertex $x$ of $G$, $x \leq_{0} x$
- for $\alpha$ an ordinal, $x \leq_{\alpha} y$ iff for all $u$ in $N[x]$, there is a $v$ in $N[y]$ such that $u \leq_{\beta} v$ for some ordinal $\beta < \alpha$
- let $\rho$ be least ordinal such that $\leq_{\rho} = \leq_{\rho+1}$ (exists as there are $2^{|V(G)|^2}$ many binary relations on $G$)
  - let $\leq = \leq_{\rho}$
- $G$ is cop-win iff $x \leq y$ for all $x, y$ in $V(G)$

- difficult to apply above in practice to determine if a graph is cop-win
• the topic of infinite cop-win graphs was relatively quiet for many years
• exceptions:
  – (Polat, 96), (Chastand, Laviolette, Polat, 02): generalizations of dismantling to the infinite case
• Brightwell, Winkler circa 2000: connections between finite cop-win graphs, locally finite graphs, and physics
On cop-win graphs

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Received 14 September 2001; received in revised form 28 September 2001; accepted 15 October 2001

In memory of Martin Farber, 21 August 1951–9 August 1989

Abstract

Following a question of Anstee and Farber we investigate the possibility that all bridged graphs are cop-win. We show that infinite chordal graphs, even of diameter two, need not be cop-win and point to some interesting questions, some of which we answer.

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1. Prologue

In 1986, Martin Farber asked the first author whether the result that finite bridged graphs are cop-win extended to infinite bridged graphs of finite diameter. It took a few years to get a counterexample, the difficulty being the finite diameter. From the example, we then extracted some information about the structure of bridged graphs and produced a general construction of a class of counterexamples. Along the way, other interesting questions were raised, some of which have only recently been answered in [4].
(Hahn, Laviolette, Sauer, Woodrow, 2002)

- infinite (in all cardinalities) robber-win chordal graphs of diameter 2
  - stark contrast to the finite case
- two proofs: 1) Compactness, 2) direct construction
  - sketch of (1): there exist finite chordal diameter 2 graph with long games (cops takes at least k moves to win for given $k > 0$ assuming optimal play)
  - axiomatize “chordal, diameter 2” in 1st order language of graphs
  - add axioms asserting that “the robber can survive $k$ moves”
  - use Compactness $\square$
The infinite random graph

Theorem (Erdős, Rényi, 63): With probability 1, any two graphs sampled from $G(\mathbb{N}, p)$ are isomorphic.

- isotype $\mathbb{R}$ unique with the e.c. property:

For all finite $A$, $B$, there exists $z$
Homogeneity

• **R** is homogeneous: isomorphism between finite induced subgraphs extend to automorphism

• **R** plays a prominent role in the (Lachlan, Woodrow, 80) classification of the countable homogeneous graphs
$c(R)$ is infinite

strongly 1-e.c. : $|A|=1$, $B$ finite
Cop density

- $R$ is infinite, but is the limit of a chain of finite graphs
- for $G$ finite, define the *cop density* as
  \[ D(G) = \frac{c(G)}{|V(G)|} \]
- If $C = (G_n : n \geq 0)$ is a chain of graphs with limit $G$, then define
  \[ D_C(G) = \lim_{n \to \infty} \frac{c(G_n)}{|V(G_n)|} \]
- $D_C(G)$ is real number in $[0, 1]$ (when defined)
Example: infinite path

\[ G \]

\[ \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \ldots \]

- \( c(G) \) infinite

\[ D_c(G) = \lim_{n \to \infty} \frac{D_c(G_n)}{n} = \lim_{n \to \infty} \frac{1}{n} = 0 \]
Theorem (Bonato, Hahn, Wang, 07)

For all real numbers \( r \) in \([0,1]\), there is a chain \( C \) with \( D_C(R) = r \).

Idea of proof: 
- fix a rational sequence in \([0,1]\) with limit \( r \)
- form chain with limit \( R \); modify cop densities at each time-step by increasing cop number with isolated nodes, while increasing order (but not cop number) by adding end-vertices
- solve infinitely many (independent) Diophantine equations

\( \square \)

- theorem holds for strongly 1-e.c. graphs
A 0-1 law

- define the upper density of $G$:

$$\text{UD}(G) = \sup_{C \text{ a chain in } G} D_C(G)$$

Theorem (Bonato, Hahn, Wang, 07)

1. For all $G$, $\text{UD}(G)$ is 0 or 1.

2. $\text{UD}(G) = 1$ iff if $G$ is a spanning subgraph of $R$.

3. $\text{UD}(G) = 1$ iff $G$ is strongly 0-e.c. ($|A|=0$, $|B|$ finite)
Cop number and adjacency properties

strongly 1-e.c.

\[ \downarrow \]

\[ c(G) \text{ infinite} \]

(no arrow reverses)

\[ \downarrow \]

strongly 0-e.c.
How many cop-win graphs are there?

- **folklore/(Polat, Sabidussi, 94):** For each infinite cardinal $\kappa$, there exist $2^\kappa$ many order $\kappa$ rayless trees
  - gives so-called **large classes** of cop-win graphs

- **regular** cop-win graphs?
  - **Exercise:** a finite regular cop-win graph is a clique.
What about vertex-transitive (vt) graphs?
Large classes of vt cop-win graphs

• Theorem (Bonato, Hahn, Tardif, 08) There are large classes of vertex-transitive cop-win graphs.
  – stark contrast with the finite case

• sketch of proof: use strong products

\[ \bigotimes_{i \in I} G_i \]

\[ V(\bigotimes_{i \in I} G_i) = \{ f : I \to \bigcup_{i \in I} V(G_i) : f(i) \in V(G_i) \text{ for all } i \in I \}, \]

\[ E(\bigotimes_{i \in I} G_i) = \{ fg : \text{ for all } i \in I, f(i) = g(i) \text{ or } f(i)g(i) \in E(G_i) \}. \]
Weak strong products

- strong products with infinitely many factors may be disconnected (eg: factors are all finite paths)
- weak strong product: fix a vertex $f$ (the base); only allow vertices that differ in finitely many factors from $f$
- weak strong product is connected and order $\kappa$ if each factor has order $\kappa$
Paradoxical lemma

- **Lemma**: If all the factors are equal to fixed graph $H$, then we may choose the index set $I$ and base $f$ so that the weak strong product is $vt$.
  - weak strong power: $\text{WSP}(H)$
Sketch of proof

• fix tree $T$ of order $\kappa$ and add a universal vertex to form $T'$
• form $G = \text{WSP}(T')$
  – $G$ is order $\kappa$ and cop-win (put cop on base; robber caught in at most two moves)
  – $vt$ by paradoxical Lemma
• form $H(T')$:
  – $\text{vertices}$: maximal proper intersections of maximal cliques in $G$ ($K_3$ in each factor except one, which is a $K_2$)
  – $\text{edges}$: union is a maximal clique ($K_3$ in each factor)
• $H(T')$ consists of disjoint copies of $T'$ so is an isomorphism invariant.

• theorem extends to $k$-cop-win graphs for all $k > 1$
Universality

Theorem (Bonato, Hahn, Tardif, 08)

1. There are large classes of cop-win graphs $G$ whose group (monoid) embeds all groups (monoids) of at most $|V(G)|$.

2. Assuming (GCH), there are large classes of cop-win graphs $G$ which contain all graphs of order at most $|V(G)|$ as induced subgraphs.

• extends to the $k$-cop-win case, $k > 1$
The next 60 years

- **Meyniel’s conjecture**: For a connected graph $G$ of order $n$,
  \[ c(G) = O(n^{1/2}). \]

- **Structure of finite $k$-cop-win graphs, $k \geq 2$**
  - dismantling/local-type characterization?

- **Goldstein, Reingold conjecture**: computing the cop number of a given graph is EXPTIME-complete
The infinite case

From (Hahn, Laviolette, Sauer, Woodrow, 2002):

... 

Several questions are left open. The first and most obvious is that of characterizing those infinite graphs for which the Nowakowski–Winkler relation is trivial, by some other, simpler, means. In other words, can we tell which infinite graphs are cop-win?

The second group of question concerns the relationship between the length of the game and various parameters of graphs. Given a cop-win graph, what is the maximum number of moves the cop needs to win? Is there a good (i.e. achievable) bound in terms of some known parameter? We know from Section 3 that the diameter does not qualify.
1st question of HSLW: unclassifiability

- our results suggest that the answer to the first question is **no**
  - properties of graphs where cop number infinite?

- are there examples of large classes of vt graphs with **bounded chromatic number**?
  - examples using strong products contain infinite cliques…
2\textsuperscript{nd} question of HSLW: capture time

- (Bonato, Golovach, Hahn, Kratochvíl, 2008) capture time, $ct(G)$: length of cops and robber game assuming optimal play
  - if order $|V(G)| = n > 4$, then $ct(G) < n - 4$
    - examples realizing this upper bound
  - 2-dismantlable: 2 corners in each step of dismantling
    - includes chordal graphs
    - $ct(G) \leq n/2$
- finding connections between $ct(G)$ and existing graph parameters remains open
- applications to moving target search in CS
• preprints, reprints, contact:

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