# On Meyniel Extremal Families of Graphs

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# Meyniel's Conjecture

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- (Frieze, Krivelevich, Loh, 2012), (Lu, Peng, 2012), (Scott, Sudakov, 2011): $c(G) = O\left(\frac{n}{2^{(1-o(1))\sqrt{\log_2 n}}}\right)$

Let $I$ be an infinite set of positive integers.

The family $\{G_n\}_{n\in I}$ is called **Meyniel extremal** if there exists a positive constant $d$ such that for $G_n$ we have $c(G_n) \geq d\sqrt{n}$ for all $n \in I$.

# Incidence Graphs of the Projective Planes

- A **projective plane** $P$ consists of a set of points and lines obeying certain axioms
  1. There is exactly one line incident with every pair of distinct points
  2. There is exactly one point incident with every pair of distinct lines
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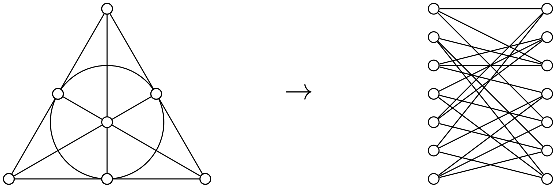
- Projective planes are known to exist for prime powers



# Incidence Graphs of the Projective Planes

If $I$ is the set of prime powers then let $G_q$ for $q \in I$ be the incidence graph of the projective plane on $q^2 + q + 1$ lines and $q^2 + q + 1$ points. Then

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**Lemma (Aigner, Fromme, 1984)**

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- So $c(G_q) \geq q+1$
- So there exists some $d > 0$ with $c(G_q) \geq d\sqrt{2(q^2+q+1)}$ for all $q \in I$

# Other Meyniel Extremal Families

Known Meyniel extremal families

- from designs (Bonato, Burgess, 2013)
- from partial affine planes (Baird, Bonato, 2012)
- polarity graphs (Bonato, Burgess, 2013)
- *t*-orbit graphs (Bonato, Burgess, 2013)
- certain Cayley graphs (Hasiri, Shinkar, 2021), (Bradshaw, Hosseini, Turcotte, 2021)

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All known Meyniel extremal families have degrees in $\Theta(|\sqrt{|V|}|)$. We find families that have much smaller minimum degree

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**Theorem (Bonato, C., Marbach, 2022+)**

*Fix $0 < \varepsilon < 1$. If $\{G_n\}_{n \in I}$ is a family of $C_4$-free graphs with degrees in $\Theta(\sqrt{n})$, then there exists $r$ pairwise nonisomorphic, Meyniel extremal, spanning families of $\{G_n\}_{n \in I}$, where*

$$r \geq 2^{d(\varepsilon)\delta(G_n)}$$

*where $d(\varepsilon)$ is a constant depending on $\varepsilon$.*

# Tools: New Lower Bound

Verification for many Meyniel extremal families rely on the following lemma (or (Aigner, Fromme, 1984))

**Lemma (Bonato, Burgess, 2013)**

*Let $t \geq 1$ be an integer. If $G$ is $K_{2,t}$-free, then $c(G) \geq \delta(G)/t$.*

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**Lemma (Bonato, Burgess, 2013)**

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We present the following generalization.

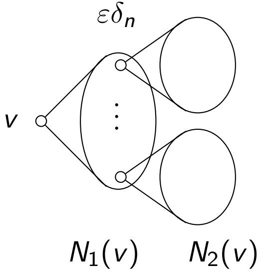
**Lemma (Bonato, C., Marbach, 2022+)**

*Let $n \geq 1, k \geq 0$ be integers with $n \geq k$. If $G$ is $K_{2,t}$-free for $t \geq 1$ an integer and has $n - k$ vertices of degree at least $D$ and $k$ vertices of degree less than $D$, with $D > k$, then*

$$c(G) \geq \frac{D-k}{t}.$$

# Sketch: New Meyniel Extremal Families

- Let $v$ be a vertex of degree $\delta_n$ in $G_n$
- Let $\mathbf{x} = (x_i)_{i=1}^{\varepsilon\delta_n}$, where $x_i \leq \deg(v_i) - 3$
- Delete $x_i$ edges from $v_i$ to $N_2(v)$
- New LB: cop number still at least $(\delta_n - \varepsilon\delta_n)/2 = \Omega(\sqrt{n})$
- Can choose many $\mathbf{x}$ that are nonisomorphic



For graphs $G$ and $H$, define the **lexicographic product** written $G \bullet H$ to have vertices $V(G) \times V(H)$, and $(u, v)$ is adjacent to $(x, y)$ if $u$ is adjacent to $x$ in $G$ or $u = x$, and $v$ is adjacent to $y$ in $H$.

Think of $G \bullet H$ as replacing each vertex $x$ of $G$ with a copy of $H$ labeled as $H_x$, such that if $xy \in E(G)$, then all edges are present between $H_x$ and $H_y$

# Regular with large chromatic number

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**Theorem (Schröder, 1998)**

*If $c(G) \geq 2$ then $c(G \bullet H) = c(G)$.*

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**Theorem (Schröder, 1998)**

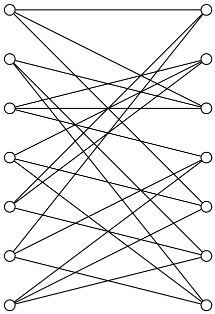
*If $c(G) \geq 2$ then $c(G \bullet H) = c(G)$.*

**Theorem (Bonato, C., Marbach, 2022+)**

*For an integer $t \geq 1$, there exist Meyniel extremal families containing graphs that are regular and with clique and chromatic number at least $t$.*

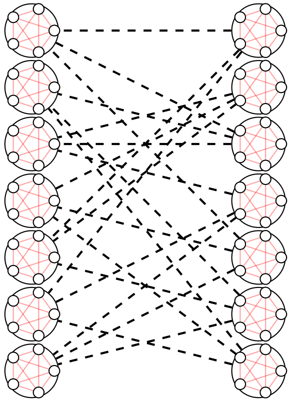
# Regular with large chromatic number

$G_n$



$\rightarrow$

$G_n \bullet K_t$



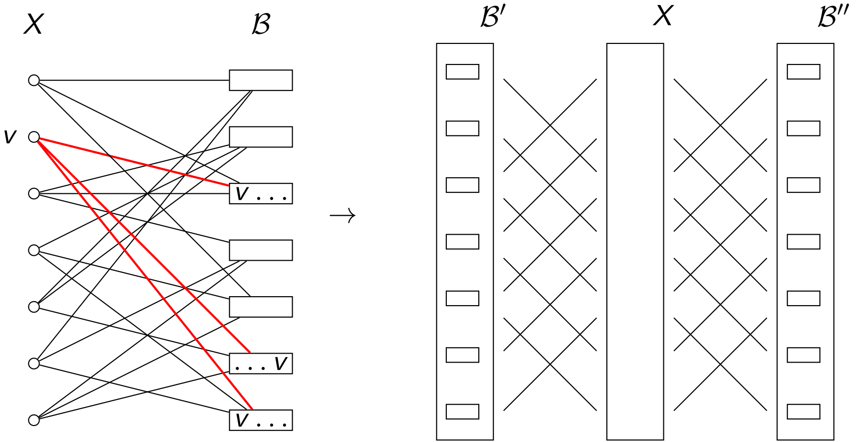
Most examples of Meyniel extremal families have very small diameter.

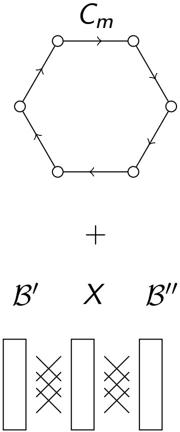
**Theorem**

*Let $q$ be a prime power and $m$ a positive integer. The graph $BF(q,m)$ is a bipartite graph with the following properties:*

1. *order $2(q^2+q+1)m$ and $(q^2+q+1)(q+1)m$ edges*
2. *$C_4$-free*
3. *diameter $2m$*
4. *$(q+1)$-regular*
5. *cop number at least $q+1$*

# Large diameter



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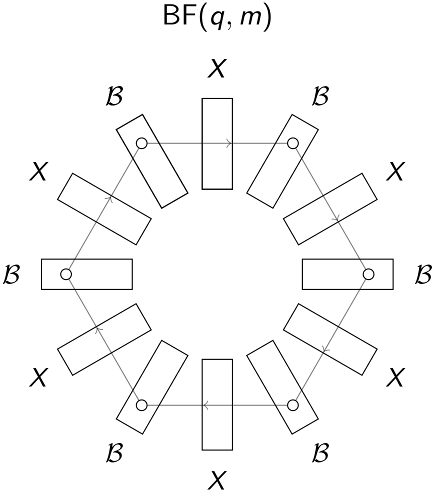

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# Tools from Hypergraphs

- A **blocking set** of a hypergraph is a subset of its vertices such that each edge contains one vertex from the subset of vertices.
- Let $x_v$ be an indicator variable for $v$ being in the blocking set.
- Integer program

$$\begin{aligned} \text{minimize} \sum_{v \in V} x_v \\ \text{subject to} \sum_{v \in e} x_v \geq 1 \end{aligned} \quad (1)$$

- If we let $x_v \in [0, 1]$, we have a linear program (2).
- (Lovász, 1975) We may approximate the solution to (1) by the solution to (2).

# Vertex transitive graphs

A graph is **vertex-transitive** if for every two vertices $x$ and $y$, there is an automorphism mapping $x$ to $y$.

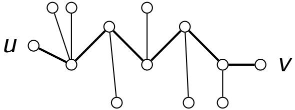
**Theorem**

*Let $G$ be a vertex-transitive graph $G$ with degree $m$, and let $d = m \cdot \mathrm{diam}(G)$. We have that*

$$c(G) \leq \frac{3n \log d}{d}$$

# Sketch

- **diameter length caterpillar (DLC)** as a minimum distance caterpillar of length diam($G$)



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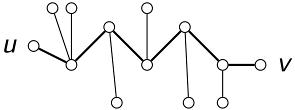


- Define a hypergraph $\mathcal{H}$ with hyperedges based on DLCs
- Vertex-transitivity allows the number of DLCs each vertex is in to be the same for all vertices

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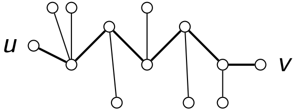


- Define a hypergraph $\mathcal{H}$ with hyperedges based on DLCs
- Vertex-transitivity allows the number of DLCs each vertex is in to be the same for all vertices
- Set up an LP to find blocking set for $\mathcal{H}$ and use (Lovász, 1975)
- Blocking set of $\mathcal{H}$ correspond to set of DLCs that cover all vertices, and 5 cops may guard each of the DLCs

# New bounds

**Corollary**

*If $G$ is a vertex-transitive graph with degree $m = \Theta(n^{1-\varepsilon})$ for a constant $0 \leq \varepsilon < 1$, then*

$$c(G) = O(n^{1-\varepsilon} \log n)$$

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# Open Problems

- There are Meyniel extremal families with $\sqrt{n}$ vertices of constant degree
- However, the maximum and average degree are still $\Omega(\sqrt{n})$
- (Hosseini et al., 2021) There are subcubic graphs with $c(G) = \Omega(n^{1/2-\varepsilon})$

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**Conjecture**

*Every Meyniel extremal family contains graphs with maximum degree $\omega(1)$*

**Conjecture**

*Every Meyniel extremal family contains graphs with average degree $\omega(1)$*

# Questions?

**THANK YOU**