Game of Cops and Robber

On Geodesic Spaces

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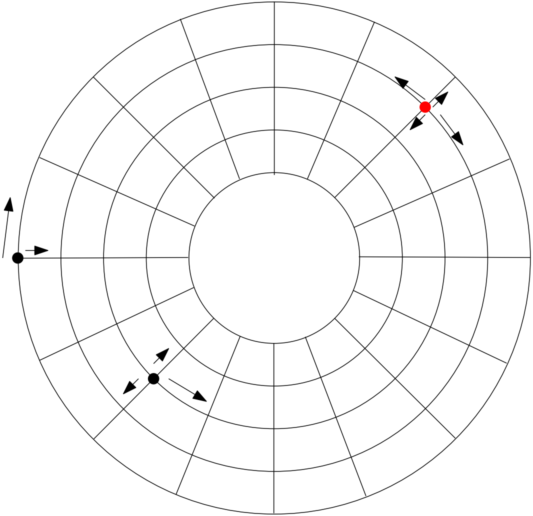
Overview

- Cops and Robber game on graphs
- Graphs on a fixed surface and Meyniel's conjecture
- Game on geodesic spaces
- Methods and examples

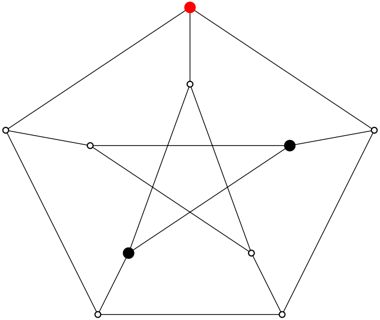
Game of Cops and Robber



Cop number of a graph

Cop number of G: $\boxed{c(G)}$

min number of cops to guarantee capturing the robber on G



Petersen's graph is the smallest graph with cop number 3.

Which graphs have small cop number

- Trees and tree-like graphs
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- d-dimensional grids $P_n \square P_m \square P_l$, or toroidal grids $C_n \square C_m \square C_l$

Large cop number (examples with $c(G) = \widetilde{\Theta}(n^{1/2})$)

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Large cop number (examples with $c(G) = \widetilde{\Theta}(n^{1/2})$)

- Graphs with min degree $\Theta(\sqrt{n})$ and girth ≥ 5
- Random graphs (Bollobás, Kun, Leader 2013; Łuczak and Prałat 2010)
- Cubic graphs (Gonzalez Hermosillo de la Maza, Hosseini, M. 2021)
- Cayley graphs of Abelian groups (Frankl 1986, Bradshaw 2018)

Graphs on a fixed surface

Theorem (Schroeder, 2001)

If G can be embedded onto a surface of genus g, then
$\boxed{c(G) \le \frac{3}{2}g + 3}$

$\frac{3}{2}$ was improved to $\frac{4}{3}$ by Erde, Lehner, Pitz, Bowler (2019)

Theorem (Erde and Lehner (2021+))

If G can be embedded onto a surface of genus g, then
$\boxed{c(G) \le \alpha\, g + 3}$*, where* $\alpha \approx 1.27 + o_g(1)$.

Graphs on a fixed surface. II

$c(g) := \max\{c(G) \mid G \to S_g\}$

Conjecture (Schroeder, 2001)

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Conjecture (Mohar)

$$\boxed{c(g) = \widetilde{\Theta}\left(g^{1/2}\right)}$$

Reason?

- Lack of examples coming close to $g + 3$
- Relationship to another conjecture

Meyniel Conjecture. I

There are many examples of n-vertex graphs, whose cop number is of order $\sqrt{n}$.

Conjecture (Meyniel (1985))

There is a constant α such that every graph of order n has $\boxed{c(G) \leq \alpha\sqrt{n}}$.

Theorem

$c(G) \leq O\left(\frac{n}{2^{(1-o(1))\sqrt{\log n}}}\right)$

Scott and Sudakov (2011), Lu and Peng (2012), Frieze, Krivelevich and Loh (2012).

Meyniel Conjecture. II

Conjecture (Weak Meyniel Conjecture)

There are positive constants α and ε such that every graph of order n has $\boxed{c(G) \leq \alpha\, n^{1-\varepsilon}}$

Theorem (Gonzalez Hermosillo de la Maza, Hosseini, M. 2021)

If all subcubic graphs satisfy the ε-Weak Meyniel conjecture, then ALL graphs satisfy the $\frac{1}{2}\varepsilon$-Weak Meyniel conjecture.

Meyniel Conjecture. III

Game can be defined on metric graphs $X(G, w)$
where edges are intervals of length $w(e)$, $e \in E(G)$

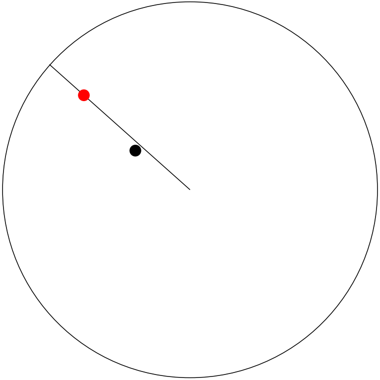
Players move "continuously" with unit speed along the edges of G

Edge-contraction: $w(e) \to 0$
Edge-deletion: $w(e) \to \infty$

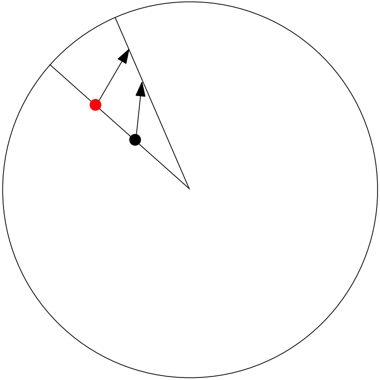
Conjecture (M. 2021)
For every n: $\boxed{\sup_w c(K_n, w) = \Theta\left(\sqrt{n}\right)}$

Lion and Man

Richard Rado / Littlewood / Bollobás



Besicovich strategy of escaping



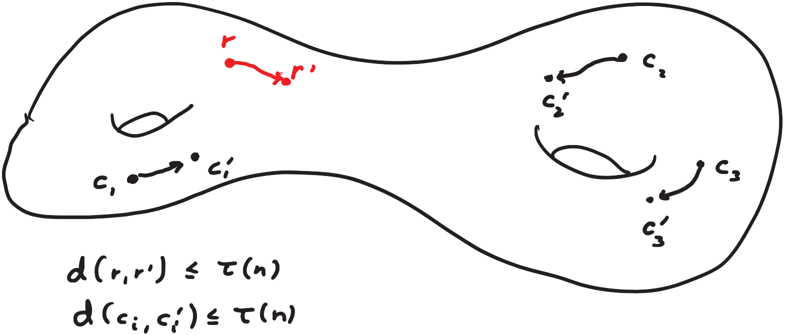
Radial strategy for approaching

Cops and Robber game on geodesic spaces

Geodesic space X: compact metric space s.t.
$\forall x, y \in X : \exists(x, y)$-path of length $d(x, y)$

Isometric path / geodesic

Robber selects agility function: $\tau : \mathbb{N} \to \mathbb{R}_+$ s.t. $\sum_n \tau(n) = \infty$



Rules of the game on X

- The robber selects initial positions of all players
- The robber selects the agility function $\tau(n)$, $n \in \mathbb{N}$
- In the n^{th} step, robber moves at most $\tau(n)$ away, then the cops move at most $\tau(n)$ away from their current positions.

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Value of the gameplay: $\inf_n d(R_n, \mathcal{C}_n)$.

Winning the game (cops) $c(X)$: $\inf_n d(R_n, \mathcal{C}_n) = 0$
Catching the robber $c_0(X)$: $\exists n : d(R_n, \mathcal{C}_n) = 0$

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Summary

- There is a min-max (inf-sup) theorem.
- Winning strategy: For each $\varepsilon > 0$, win the ε-approaching game
- Define $c_\varepsilon(X)$: the number of cops needed to win the ε-approaching game
- Then $c(X) = \sup c_\varepsilon(X)$
- It may happen that $c(X) = 1$ and $c_0(X) = \infty$

Guarding an isometric path

Basic idea of Aigner and Fromme for guarding an isometric path works:

Lemma. If P is a geodesic in X then one cop can guard it. (If R ever steps on P, he will be caught).

Proof idea: Let P be an (a, b)-geodesic. Define the shadow $\sigma(x) = b$ if $d(a, x) > d(a, b)$, and let $\sigma(x) = z$ s.t. $d(a, x) = d(a, z)$, otherwise. This is a 1-Lipschitz mapping: $d(\sigma(x), \sigma(x') \leq d(x, x')$.

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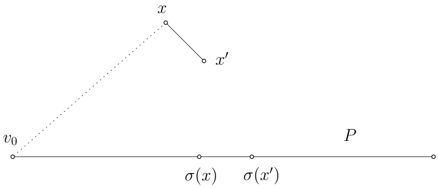
First, a cop catches the shadow $\sigma(R)$ of R.
Second, the cop maintains being at the vertex $\sigma(R)$, he can guard P.

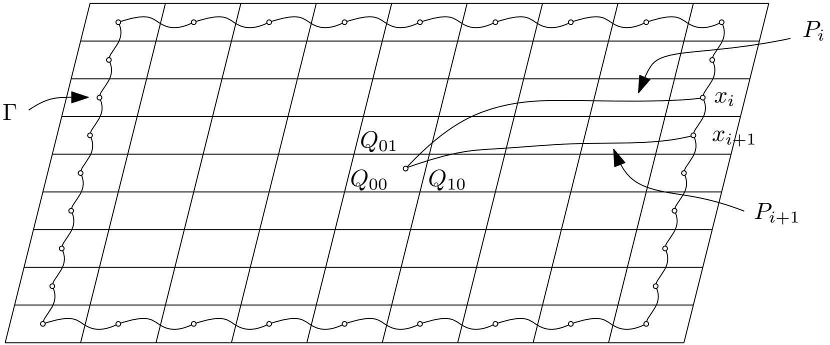


How to catch the robber on a torus

Theorem

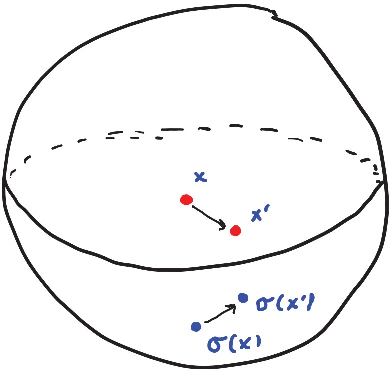
If a compact geodesic space X is homeomorphic to a subset of the torus, then $\boxed{c(G) \leq 3}$

Florian Lehner's trick:



How to catch the robber on the n-sphere

n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$



$c(S^n) = 2$ (coming ε-close)

$c_0(S^n) = n + 1$ (catching)

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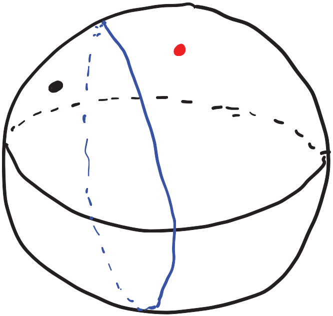
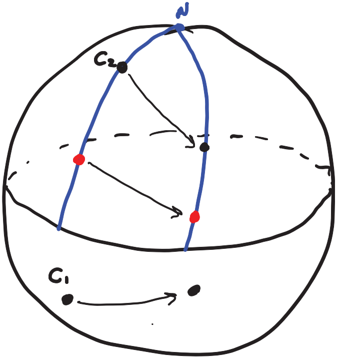


Guarding the equator



The radial strategy

Hypergraphs and simplicial complexes

$K_n^{(3)}$: the complete 3-uniform hypergraph, simplicial complex by having a simplex for each triple $\{x, y, z\} \subseteq [n]$

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Theorem. $\boxed{c_0(K_n^{(3)}) \leq 10}$

Thank you for listening!