The node cop-win reliability of a graph

Ben Cameron (he/him)

The King's University

ben.cameron@kingsu.ca

(Joint work with Maimoonah Ahmed)

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1. Intro
2. Unicyclic
3. Bicyclic
4. Conclusion

Question: How can we find the most cop-win graph is a given family?

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Result: A graph polynomial to quantify the “cop-win-ness” of a given graph.

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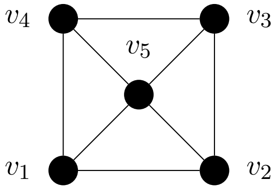

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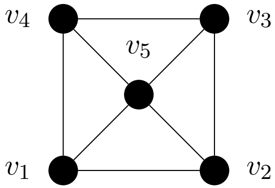

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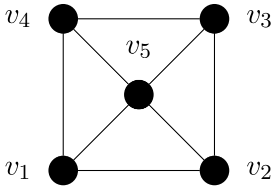

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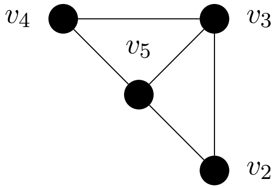

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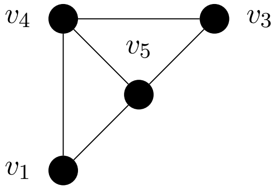

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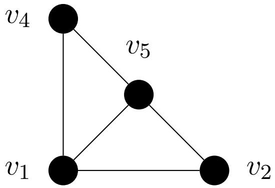

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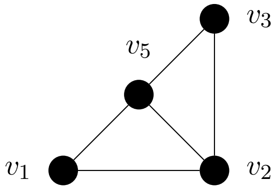

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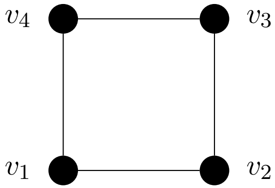

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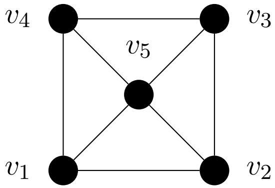

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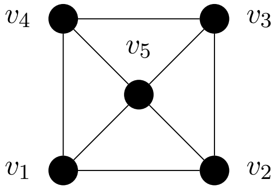

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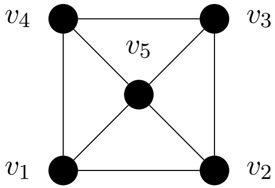

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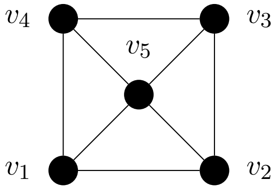

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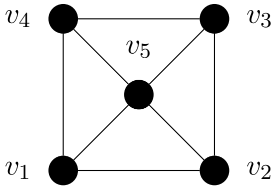

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Prob$= 5\,p\,(1-p)^4+8\,p^2\,(1-p)^3+10\,p^3\,(1-p)^2+4\,p^4\,(1-p)+p^5$

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We can now compare graphs by how cop-win they are!

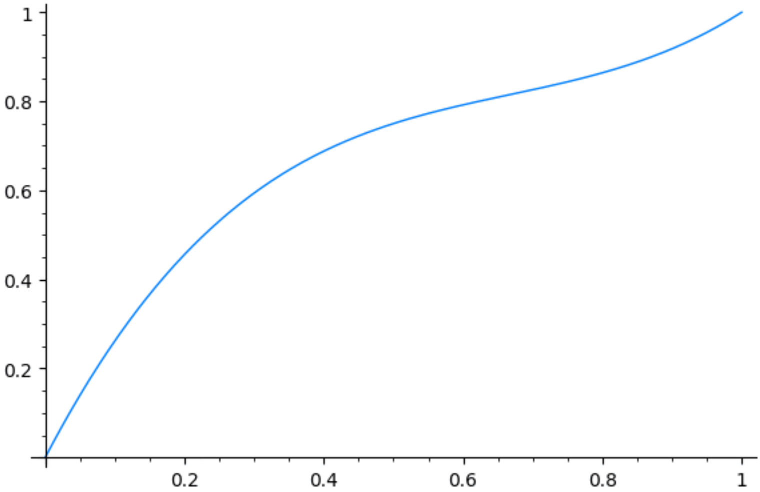

Figure: $\mathrm{NCRel}(P_n, p)$ for $3 \leq n \leq 11$.

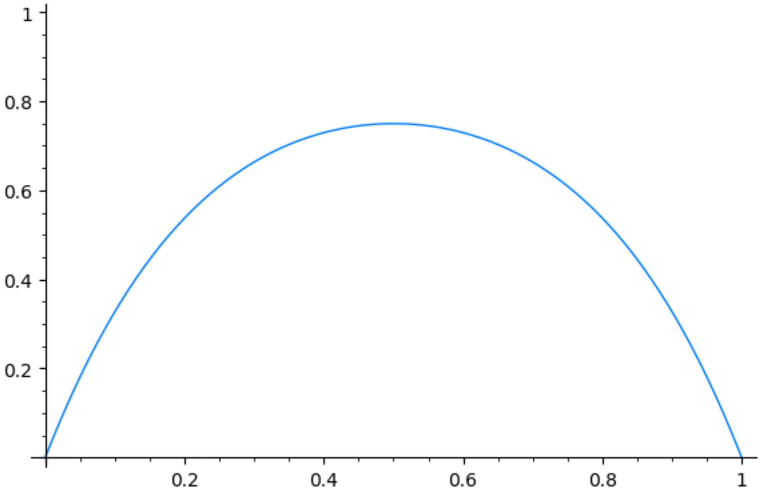

Figure: $\mathrm{NCRel}(C_n, p)$ for $4 \leq n \leq 12$.

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Question: Which families of graphs is it of interest to find UMR graph(s)?

- Let $\mathcal{U}_n$ denote the family of all unicyclic graphs of order n (connected simple graphs with n edges).

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- Known that $K_{1,n-1}$ is UMR for trees (Stivaros 1990). Unicyclic graphs next smallest interesting case.
- UMR graphs with respect to node reliability do not exist in $\mathcal{U}_n$ (Stivaros 1990).

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From computations, there is either no UMR graph or it is between U_n and C_n.

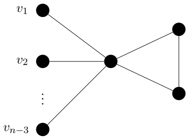

Figure: U_n.

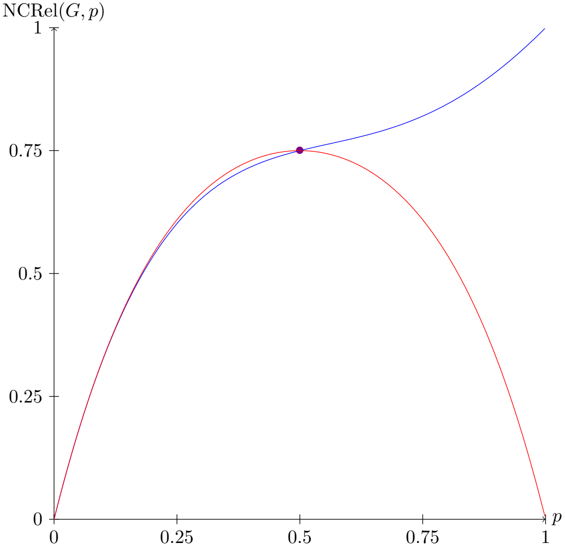

Figure: Plots of NCRel(U_4, p) and NCRel(C_4, p).

Lemma (Ahmed-C. 2022) $\mathrm{CW}(H,x) \preceq \mathrm{CW}(U_n,x)$ for all $H \in \mathcal{U}_n \setminus \{C_n\}$ and $n \geq 5$.

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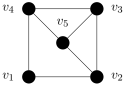
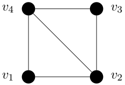
Lemma (Ahmed-C. 2022): If $v \in V(G)$ and $u \in V(H)$ such that

1) $\mathrm{CS}(G-v,x) \preceq \mathrm{CS}(H-u,x)$,
2) $\mathrm{CS}(G/v,x) \preceq \mathrm{CS}(H/u,x)$, and
3) $\mathrm{CS}(H-N[u],x) \preceq \mathrm{CS}(G-N[v],x)$,

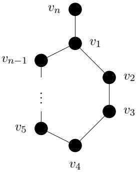
then $\mathrm{CS}(G,x) \preceq \mathrm{CS}(H,x)$.



(a) G



(b) G/v_5



(c) A_n

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Theorem (Ahmed-C. 2022): For all $n \geq 5$ U_n is UMR in $\mathcal{U}_n$.

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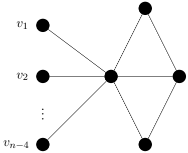


Figure: B_n.

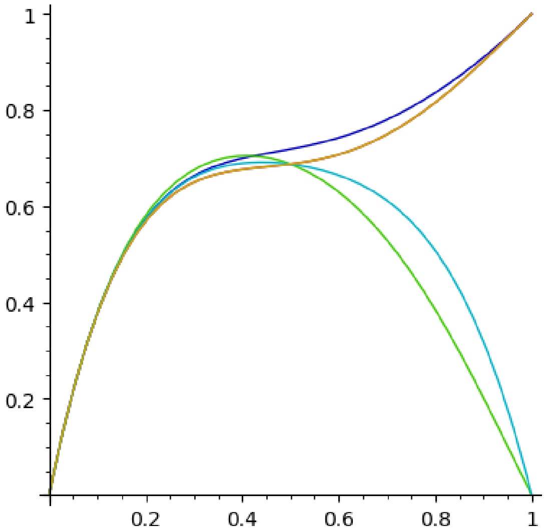


Figure: $\mathrm{NCRel}(G, p)$ for all $G \in \mathcal{B}_5$.

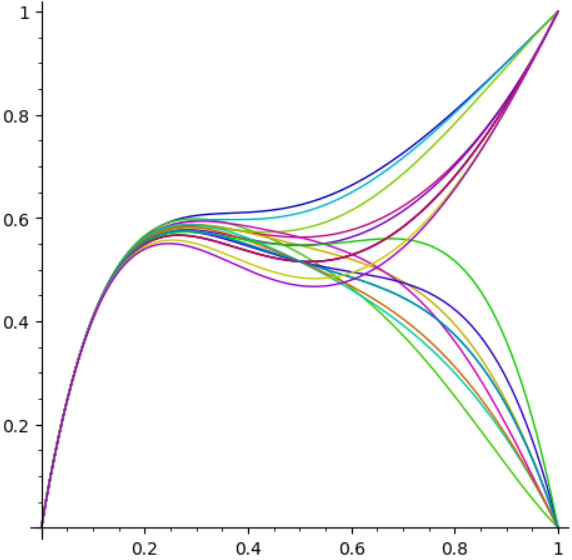
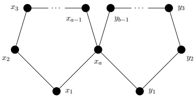
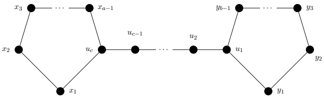


Figure: $\mathrm{NCRel}(G, p)$ for all $G \in \mathcal{B}_6$.

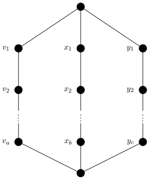
Theorem (Ahmed-C. 2022): For all $n \geq 7$, $\mathrm{CW}(H, x) \preceq \mathrm{CW}(B_n, x)$ for all $H \in \mathcal{B}_n$, therefore B_n is UMR in $\mathcal{B}_n$.



(a) $G_1(a, b)$



(b) $G_2(a, b, c)$



(c) $G_3(a, b, c)$

Figure: The bicyclic graphs $G_1(a, b)$, $G_2(a, b, c)$, and $G_3(a, b, c)$

Conjecture (Ahmed-C. 2022): For all $n \geq 2(m+1)+1$, $H_{n,m}$ is UMR in the family of m-cyclic graphs.

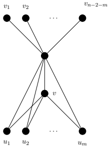


Figure: The graph $H_{n,m}$.

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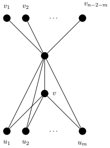

Figure: The graph $H_{n,m}$.

Question: Let $\mathcal{T}_n$ be the set of all 2-cop-win graphs of order n. Is there are UMR graph in $\mathcal{T}_n$? If so, which one?

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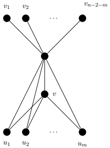

Figure: The graph $H_{n,m}$.

Question: Let $\mathcal{T}_n$ be the set of all 2-cop-win graphs of order n. Is there are UMR graph in $\mathcal{T}_n$? If so, which one?

Open Problem: Consider edge cop-win reliability.

Thank You!

Figure: Scan QR code for the paper.