Algorithms for Burning Graph Families

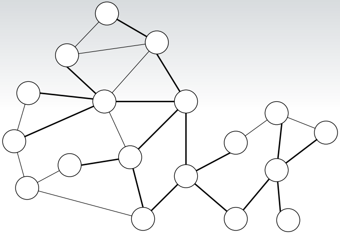
Shahin Kamali

August 6, 2021

Graph Searching in Canada (GRASCan) Workshop

Graph Burning Problem

- Given an undirected graph G, the goal is to **burn** in a minimum number of **rounds** [Bonato et al., 2014].



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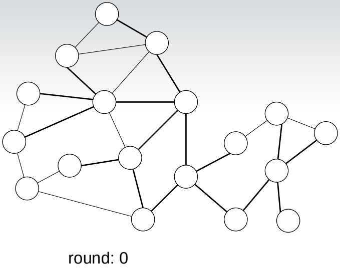

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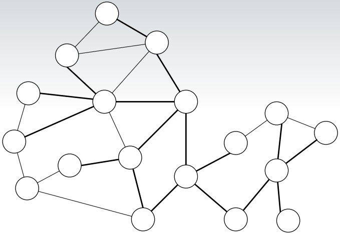

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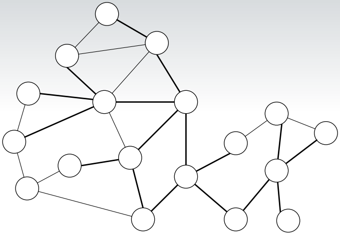

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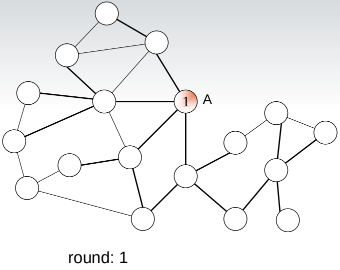

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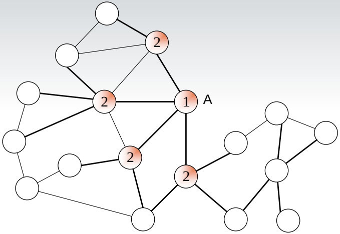

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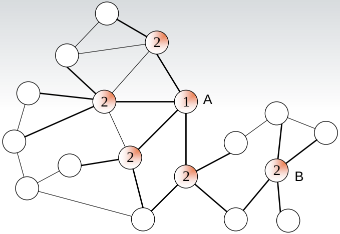

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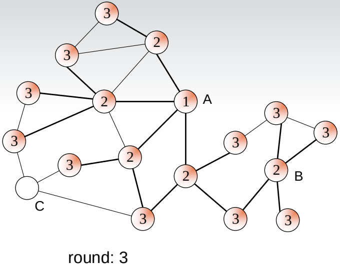

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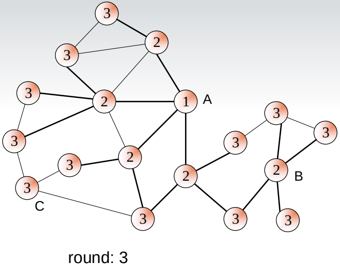

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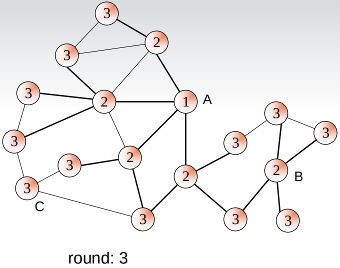

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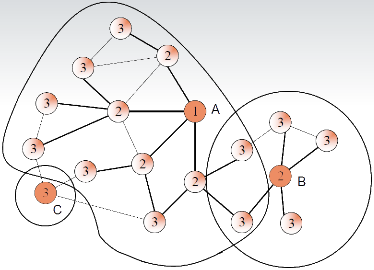
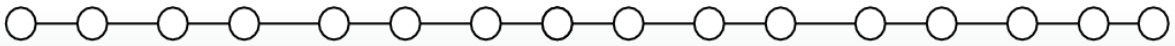


- Decision problem:
  - Can we burn G in k rounds?
  - Equivalently, can we cover the graph with "disks" of radii $0, 1, 2, \ldots, k-1$?

Burning Paths

- A path P_n of length n can be covered with disks of radii $0, 1, 2, \ldots, \lceil \sqrt{n} \rceil$ [Bonato et al. 2014].



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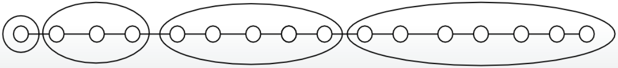

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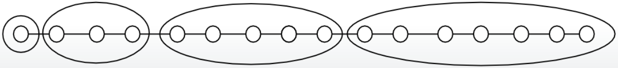

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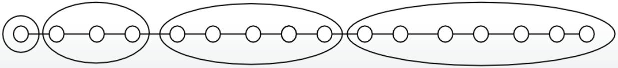

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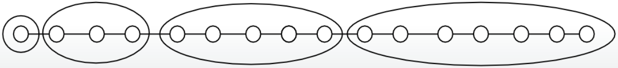
  - The burning number of any connected graph is at most $\frac{\sqrt{6}}{2}\sqrt{n} \approx 1.22\sqrt{n}$ [Land and Lu, 2016].



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  - The problem remains NP-hard for disjoint set of paths, trees, other graph families.
  - The problem is more "interesting" when the underlying graphs are sparse.
- It is claimed that the problem is APX-hard [Mondal et al., 2021] (no $(1+\epsilon)$-approximation exists assuming $P \neq NP$).

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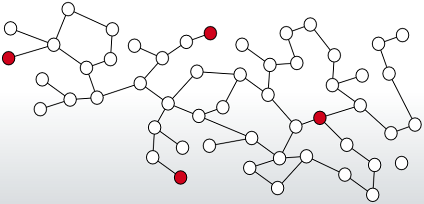

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- Example: suppose there are $r = 4$ vertices of pairwise $2r - 1 = 7$ in a graph G.
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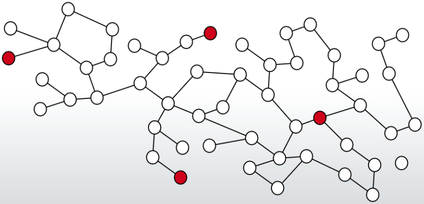

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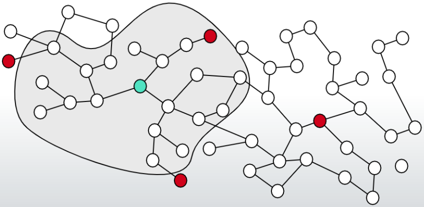

Constant Approximation Algorithm

- Define a procedure Burn-Guess(G,g) which returns:
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  - Or 'Bad-Guess', which guarantees burning cannot be complete in $g - 1$ rounds.
- The smallest value of g^* for which Burn-Guess returns a schedule gives a burning scheme that completes in $3g^* - 3$ while the optimal schedule will require $g^* - 1$ rounds to complete.
  - Approximation ratio of at most 3.

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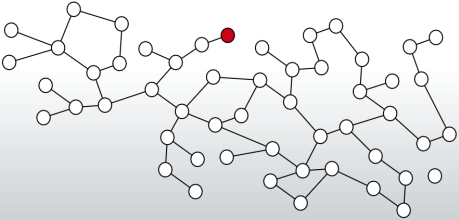

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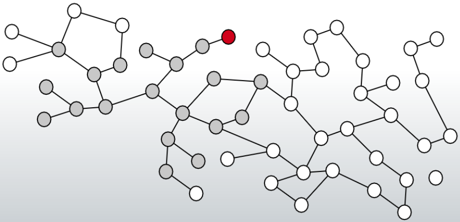

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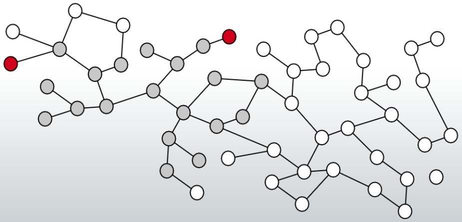

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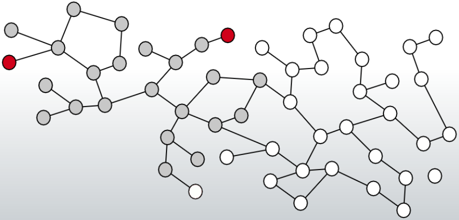

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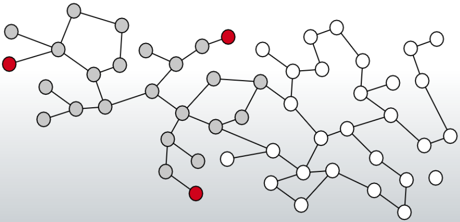

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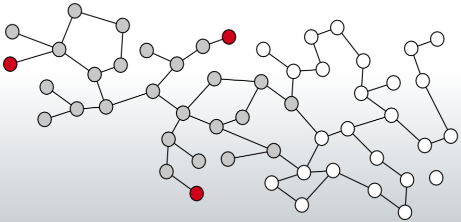

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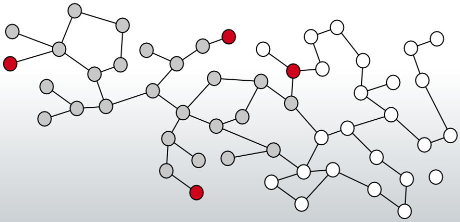

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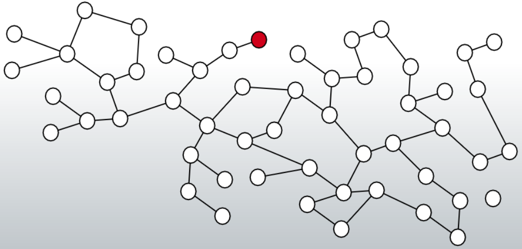

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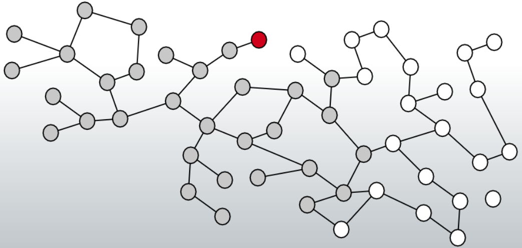

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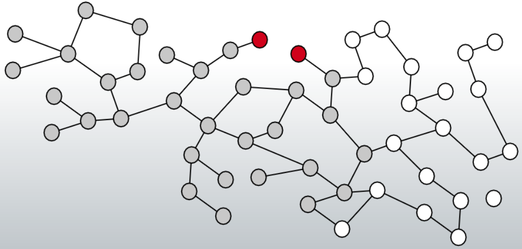

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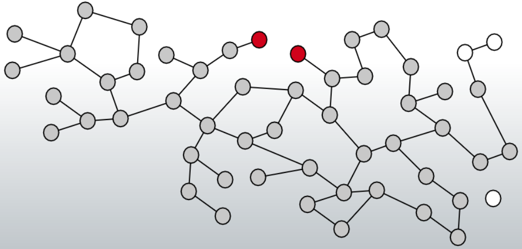

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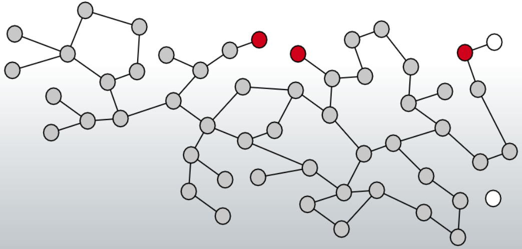

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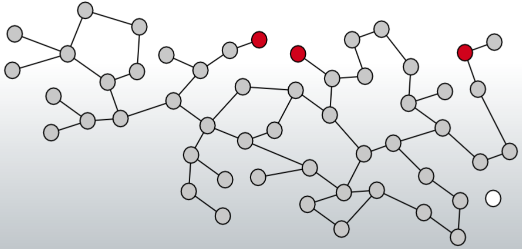

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  - If all vertices are added to L, return an arbitrary ordering of centers as the burning scheme (which completes in at most $(g-1) + (2g-2) = 3g - 3$ rounds).
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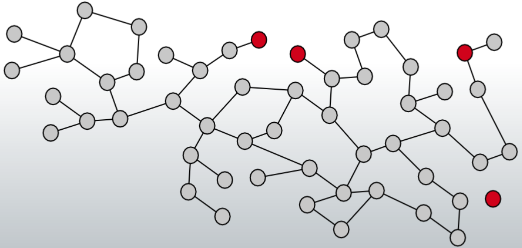

General Graph Summary

Theorem

There is a polynomial algorithm with approximation ratio of 3 *for burning any graph* $G = (V, E)$ *[Bonato & S.K., 2019].*

- What about graph families? can we get better approximation ratio for families of graphs?

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- It is possible to achieve an approximation factor of 2.
- Burn-Guess-Tree (τ, g) returns either a schedule that completes in at most $2g - 2$ rounds or 'Bad-Guess', which means burning cannot complete in $g - 1$ rounds.
- An approximation factor of at most 2 is guaranteed:
  - The schedule returned by the smallest value of $g = g^*$ completes in $2g^* - 2$ rounds.
  - For $g^* - 1$, Bad-Guess is returned, which implies that the optimal scheme requires at least $g^* - 1$ rounds.

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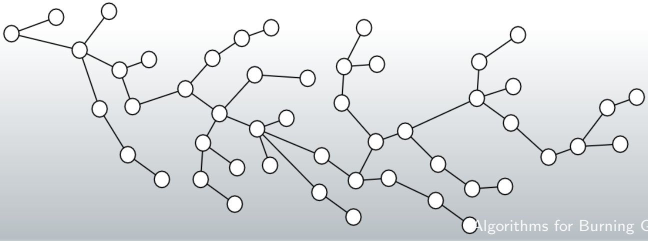

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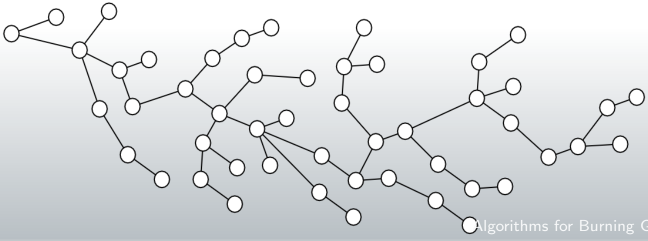

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  - Take the deepest unlabeled node x, add x to T.
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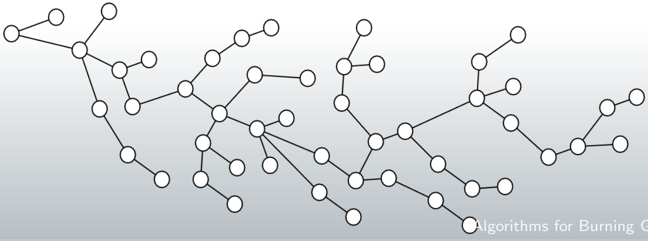

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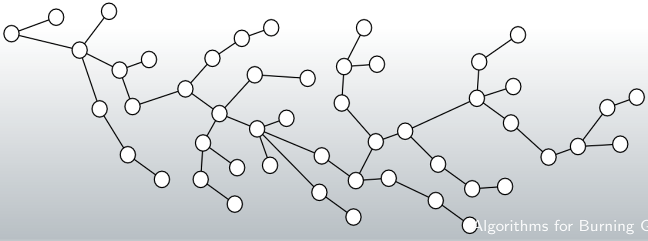

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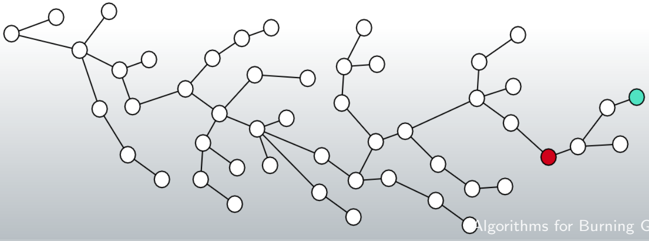

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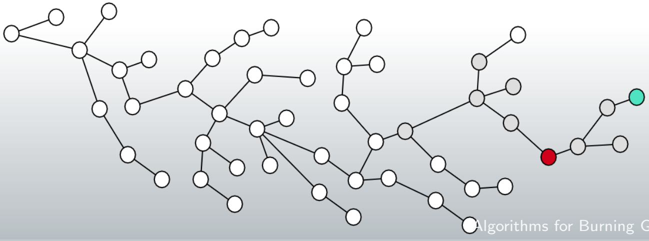

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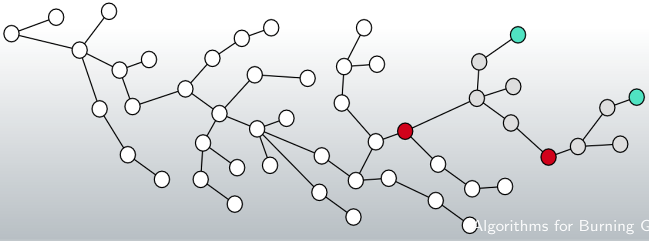

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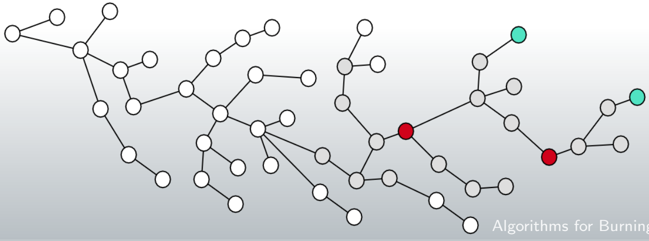

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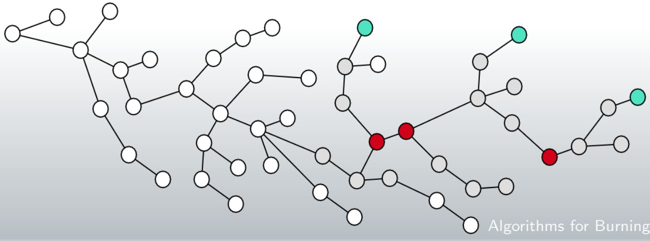

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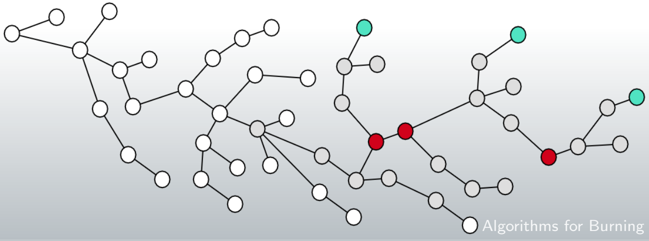

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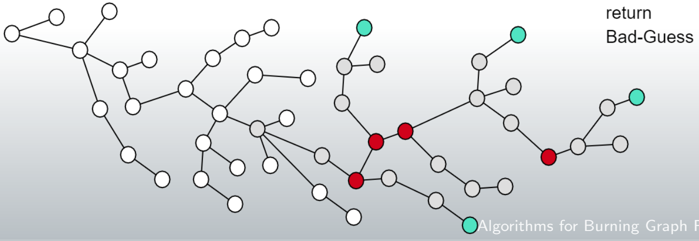

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  - When $|T| = g$, return Bad-Guess.
  - When all vertices are labeled, return any ordering of C as the burning schedule. All nodes are within distance $g-1$ of g centers.
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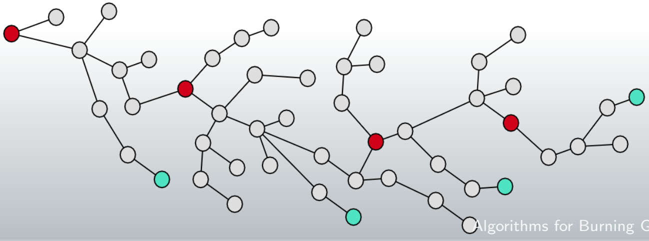

Burning Trees Summary

Theorem

There is a polynomial algorithm with approximation ratio of 2 *for burning any tree [Bonato & S.K., 2019].*

- Open question: what is the best approximation factor attainable for trees? is it possible to get an PTAS (with approximation factor $1+\epsilon$)?

Burning Cacti

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- Therefore, an approximation factor of at most 2.75 is guaranteed.
  - The schedule returned by the smallest value of $g = g^*$ completes in $2.75g^* - 2$ rounds.
  - For $g^* - 1$, Bad-Guess is returned, which implies that the optimal scheme requires at least $g^* - 1$ rounds.

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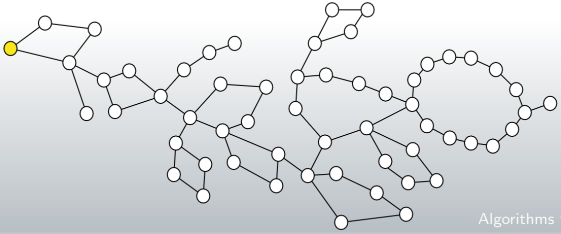

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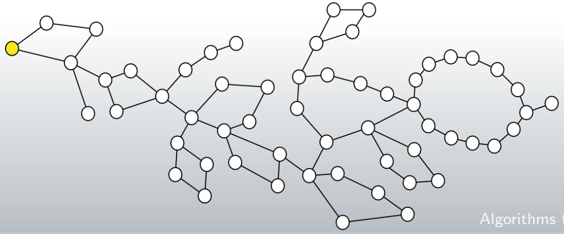

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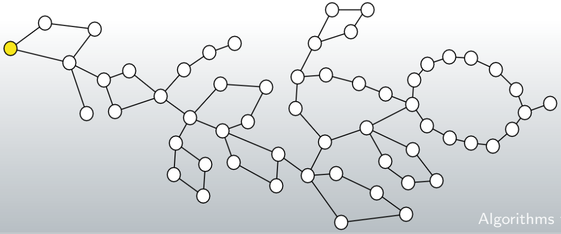

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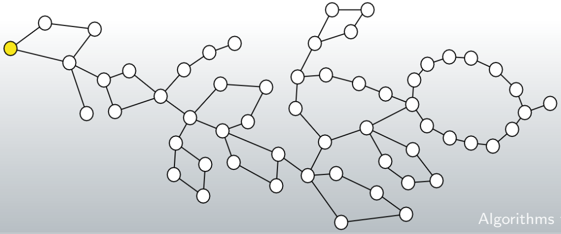

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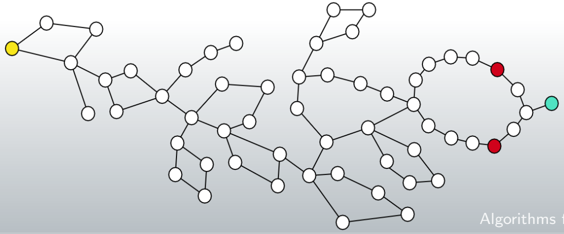

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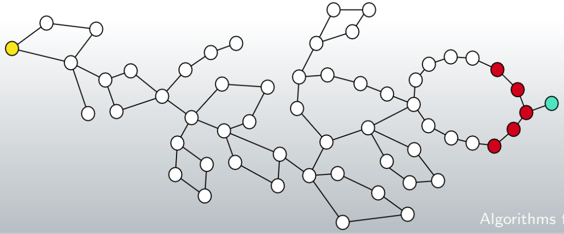

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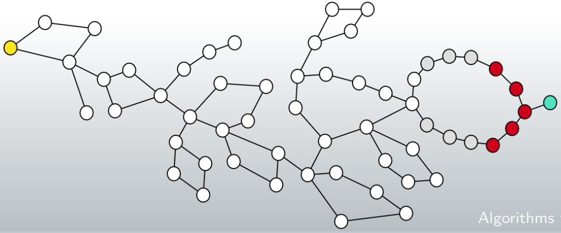

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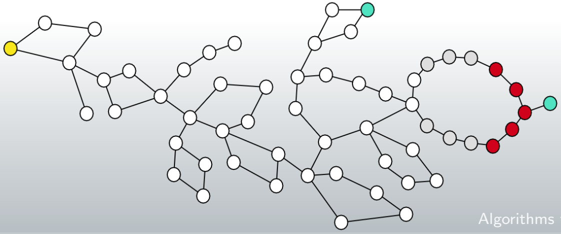

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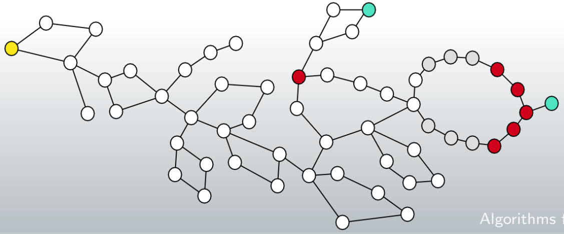

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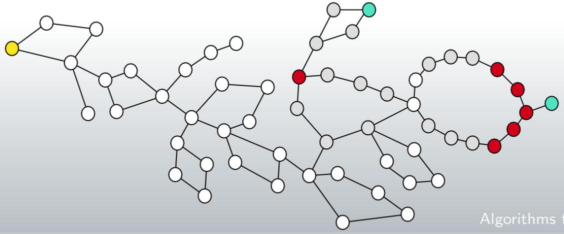

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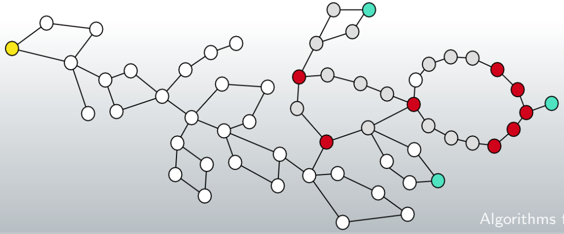

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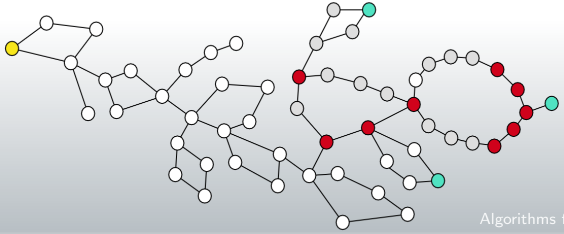

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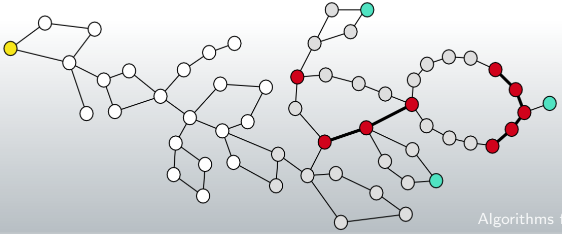

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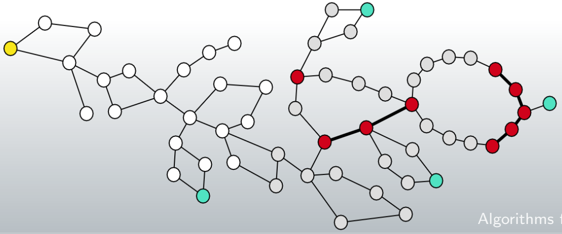

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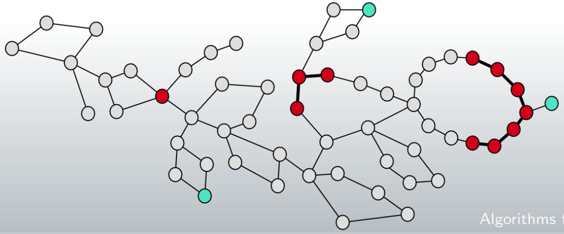
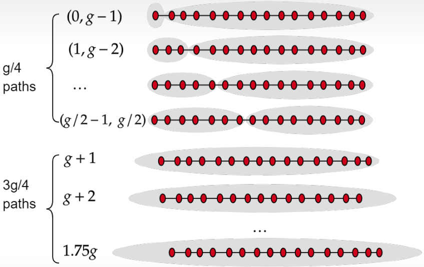


Burning Cacti

- It is possible to burn a forest C of g disjoint paths, each of length at most $2g$ nodes in at most $1.75g$ rounds.



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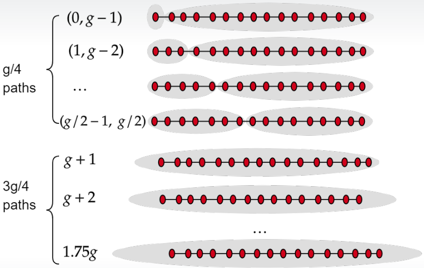

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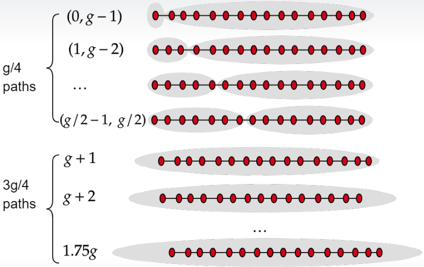
- It is possible to burn all vertices in C in $1.75g$.
- All nodes are within distance g of one of the centers, so all vertices are burned in $1.75g + g = 2.75g$ rounds.



Burning Cacti Summary

Theorem

There is a polynomial algorithm with approximation ratio of 2.75 *for burning any cactus graph [S.K. and Shabani, 2021].*

- The main idea was to burn paths of centers instead of singular centers.
- The same idea might be applied burning other graph families (e.g., Series-Parallel graphs).

Forests of Disjoint Paths

- The burning problem is NP-hard when the input graph is a forest of disjoint paths [Bessy et al., 2017].
  - Given disks of radii $0, 1, \ldots, k-1$, it is not clear which disk should be assigned to which path.
- If there are $\Theta(1)$ disjoint paths, there is a polynomial-time algorithm that generates an optimal burning scheme [Bonato and S.K., 2019].
  - Apply a dynamic programming approach!

Forests of Disjoint Paths

- Given any positive value ϵ, there is a fully polynomial-time approximation algorithm (FPTAS) that generates a burning scheme that completes within a factor $1 + \epsilon$ of an optimal scheme [Bonato and S.K., 2019].

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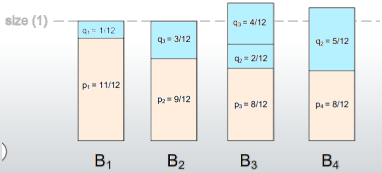
  - **Bin covering:** "cover" a maximum number of bins of unit size with a given multi-set of items with sizes in $(0, 1]$.



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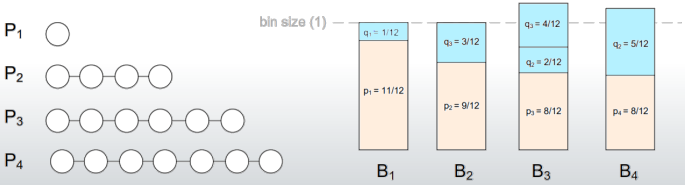
  - Think of paths as uniform "bins" that need to be "covered" by items (disks) of radii $0, 1, \ldots, k-1$.



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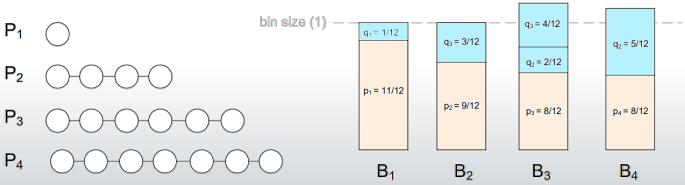
    - Think of paths as uniform "bins" that need to be "covered" by items (disks) of radii $0, 1, \ldots, k-1$.



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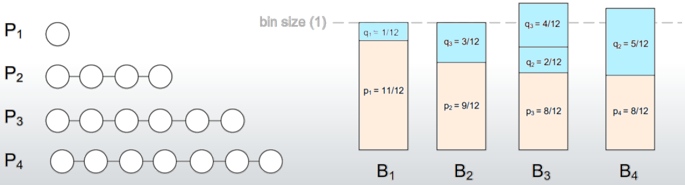

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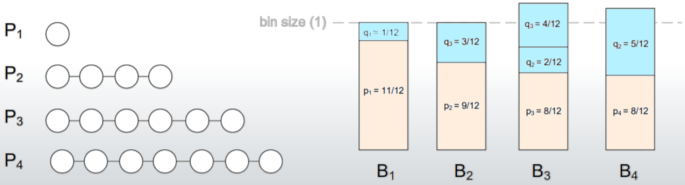

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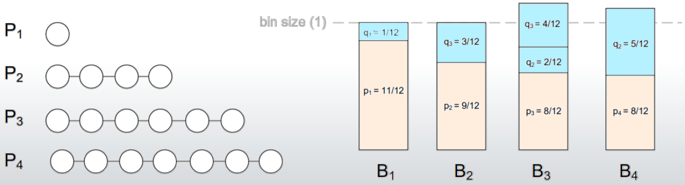
    - Items $p_1, p_2, \ldots, p_b$ project paths of various lengths into bins of unit size.



Burning Forests of Disjoint Paths Summary

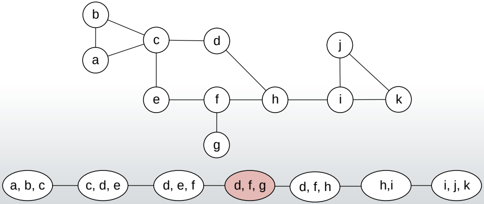
Theorem

There is a fully polynomial-time approximation scheme (FPTAS) for burning any forest of disjoint paths [Bonato and S.K., 2019].

- The complexity of the problem is settled for forests of disjoint paths.
- For what other graph families an FPTAS might be developed?

Tree Decomposition & Burning

- In a Robertson-Seymour path decomposition:
  - Path-length [Dourisboure and Gavoille, 2007] is the max **distance** of vertices in any bag.
  - The graph below has path-width 2 and path-length 3.



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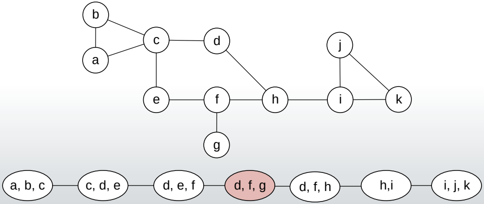

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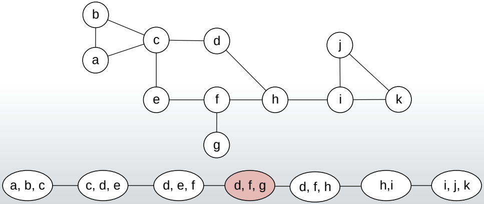

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  - A graph has path-length 1 if and only if it is an interval graph.
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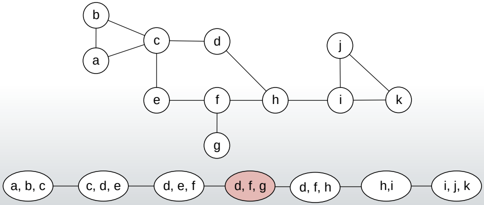

Burning of Graph Families

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Summary

Graph family	Apx. Factor	Details
general graphs	3	[Bonato and S.K., 2019]
trees	2	[Bonato and S.K., 2019]
cacti	2.75	[S.K. and Shabani, 2021]
forests of disjoint paths	$1+\epsilon$ (FPTAS)	[Bonato and S.K., 2019]
graphs of bounded path-length	$1+o(1)$	[S.K. et al., 2020]
graphs of bounded tree-length	$2+o(1)$	[S.K. et al., 2020]

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Ongoing work.