Catching an Infinite-Speed Robber on Grids

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Introduction To Cops and Robbers

- Two teams (k cops and one robber) play on a connected graph G.
- Each player occupies vertices of G.
- Cops choose their starting positions, then the robber does.
- The game is played in rounds:
  - In a round, a player may move to an adjacent vertex, or stay put.
  - If some cop occupies the robber's vertex, then the cops win. The robber wins if he is able to evade the cops forever.
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The minimum number of cops needed to win on a graph G is the cop number of G, denoted $c(G)$.

Intro To Infinite-Speed Game

What if the robber is allowed to move "faster" than the cops?

- In general, robber can be assigned speed s. He is allowed to move along a **cop-free path of length** $\leq s$ on his turn. Cops can still only move to adjacent vertices.

  (The case $s = 1$ is equivalent to the original model of Cops and Robbers.)

- We focus on variant where $s = \infty$. Robber allowed to move along any path of arbitrary length (with no cop on internal vertices).

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- The **infinite-speed cop number** of G, $c_\infty(G)$, is the minimum k such that k cops can capture a infinite-speed robber on G.

Let's Play (Infinite-Speed Game)

Let's play with two cops against an infinite-speed robber.

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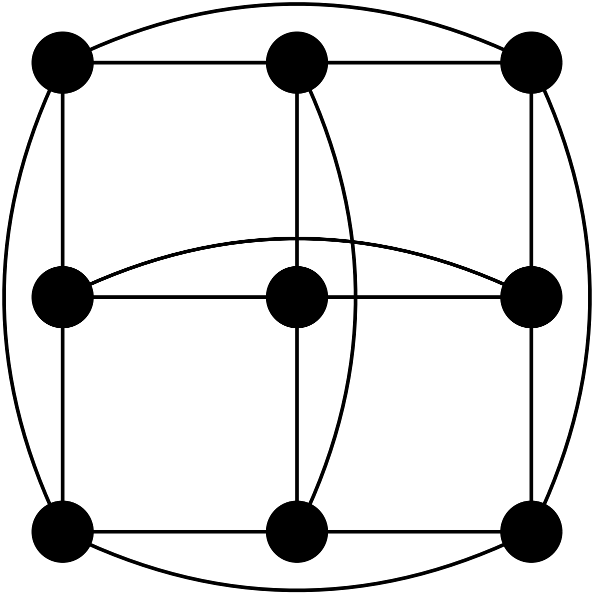

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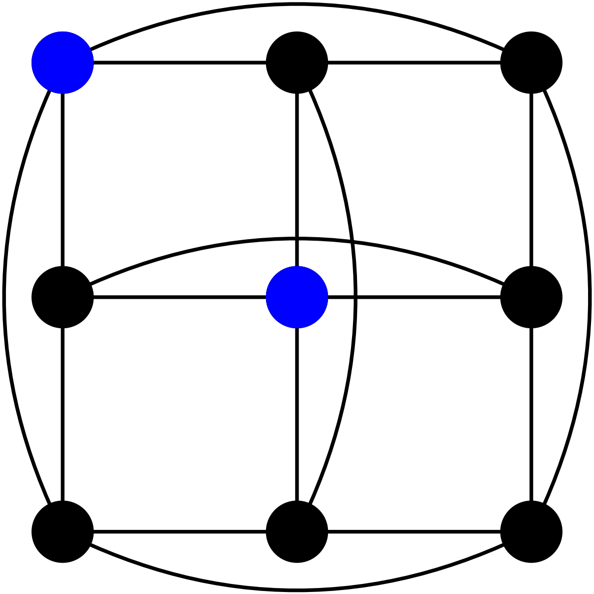

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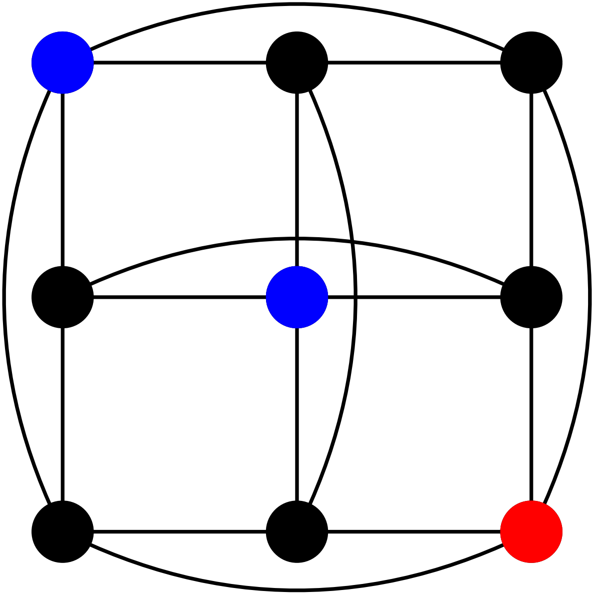

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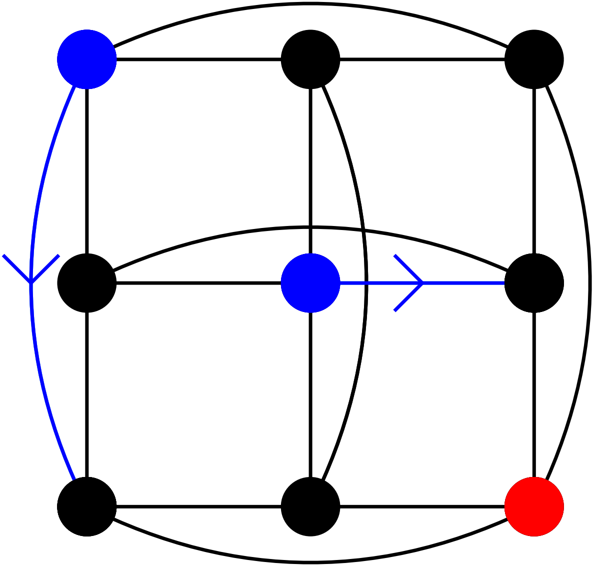

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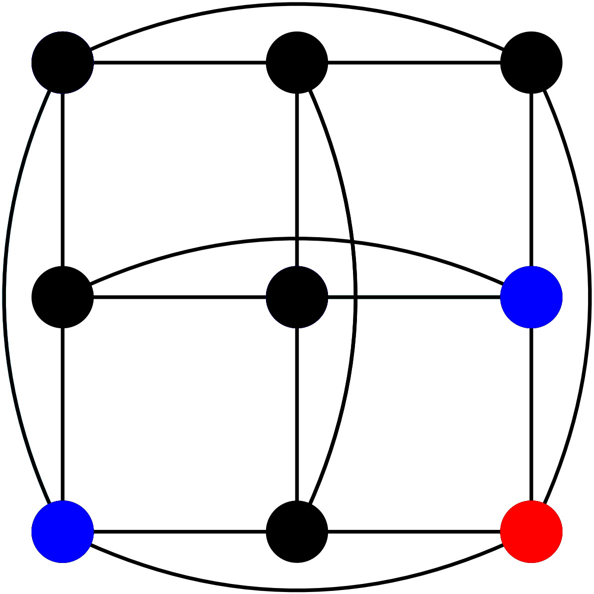

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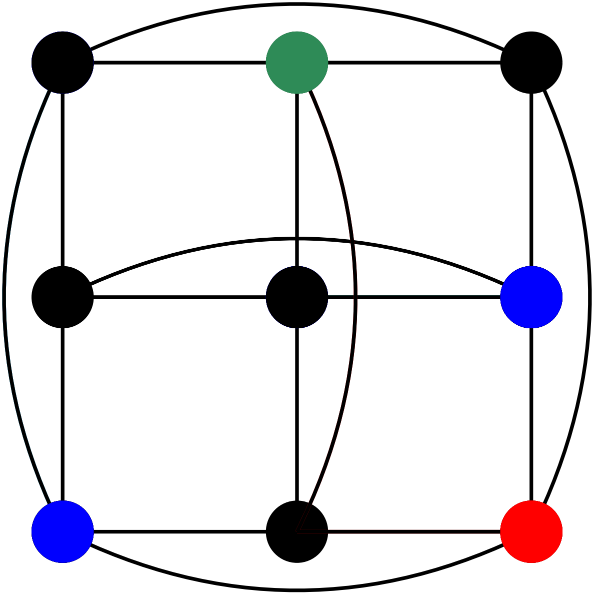

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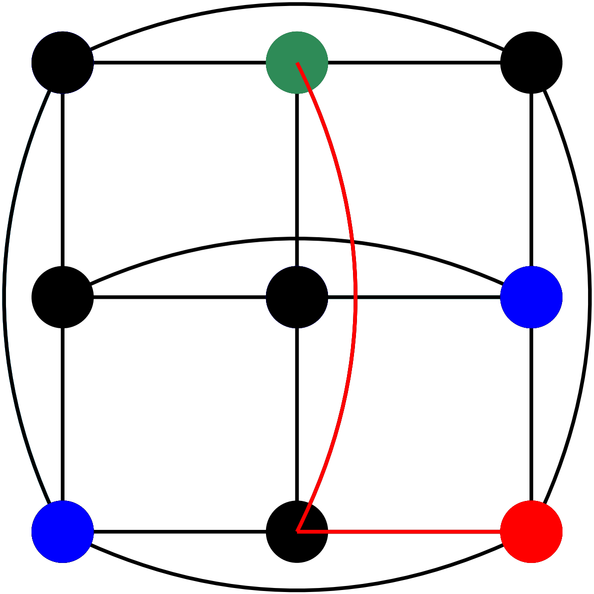

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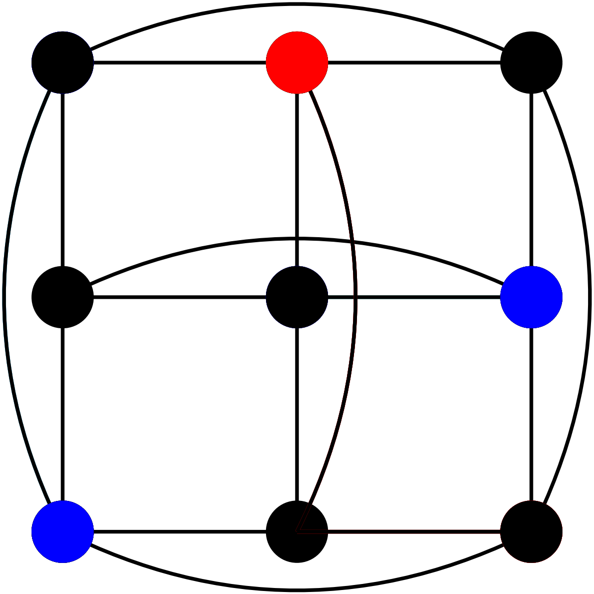

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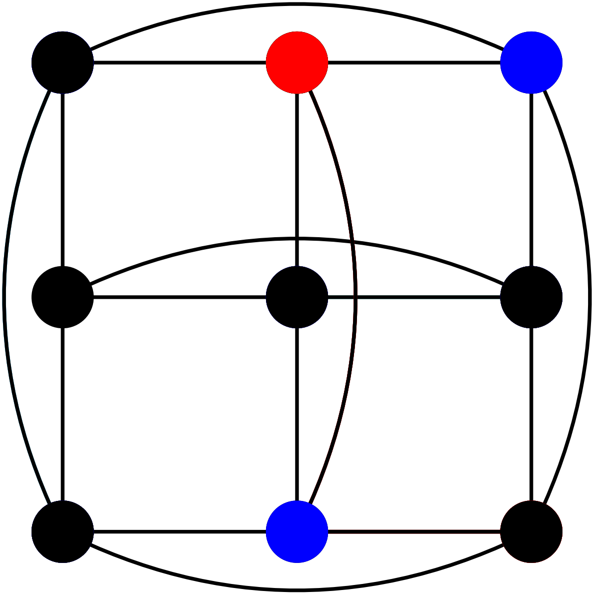

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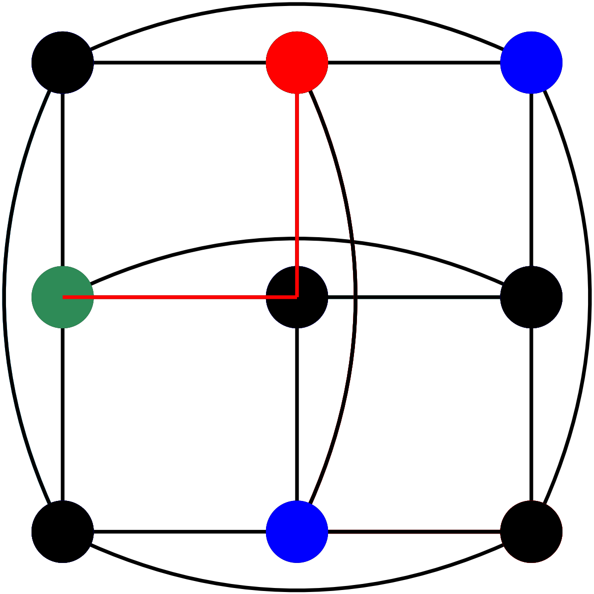

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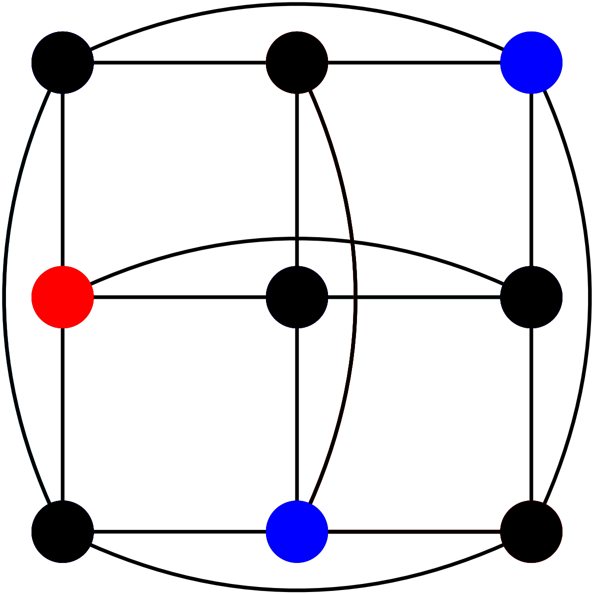

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Our focus: grids (i.e. Cartesian products of paths).

Two-dimensional grids

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Our result:

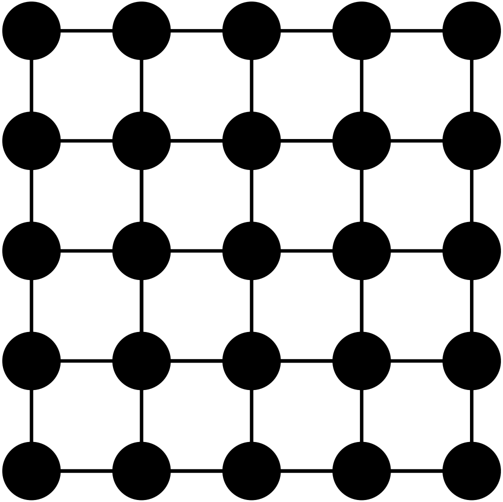
Theorem (Kinnersley, Townsend '21+)

If n is even, then $n - 1 \le c_\infty(P_n \square P_n) \le n$*;*
if n is odd and at least 3, then $c_\infty(P_n \square P_n) = n - 1$*.*

A Trivial Upper Bound

Trivial upper bound on c_∞:

- $c_\infty(P_n \square P_n) \leq n$.



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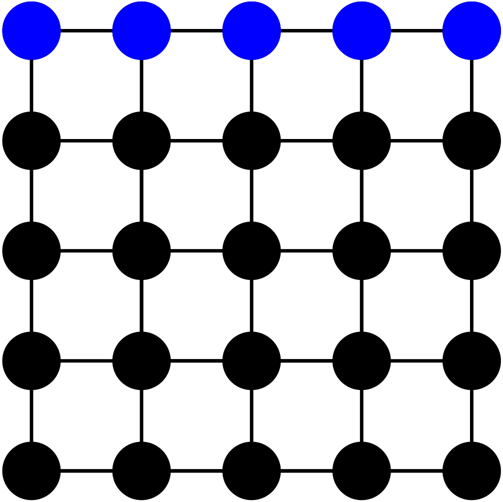

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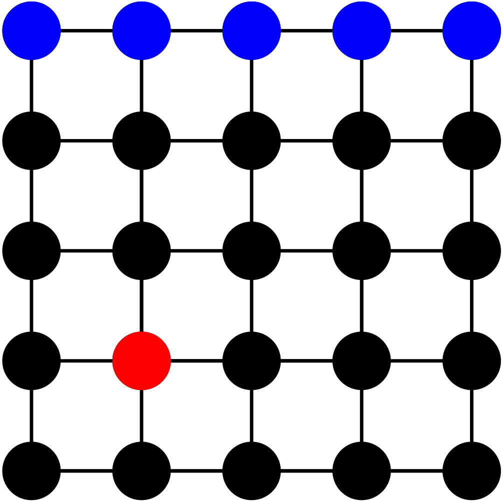

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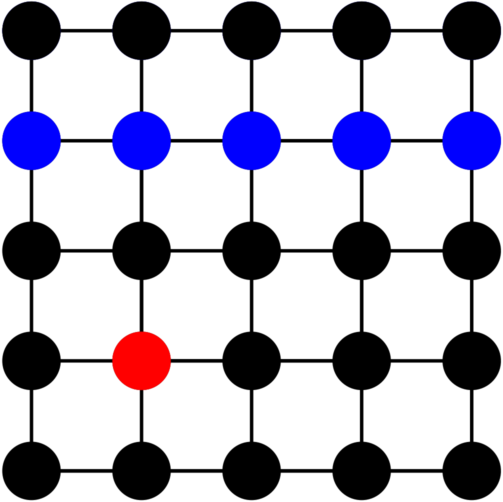

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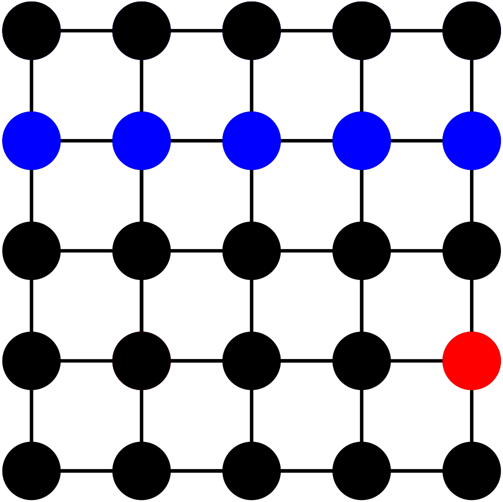

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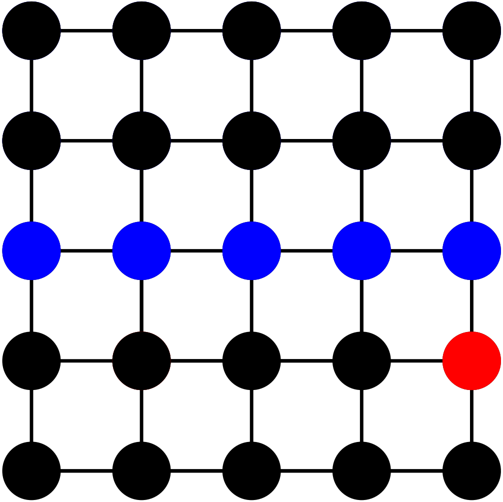

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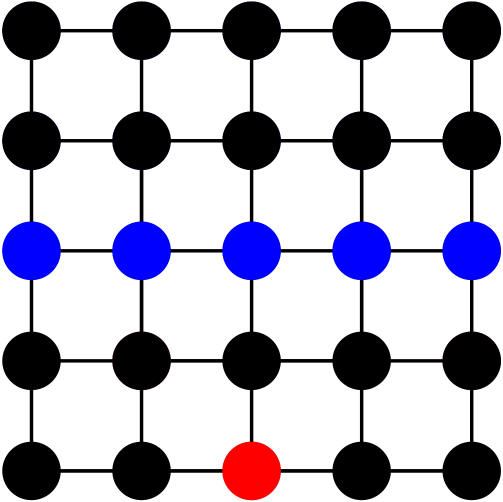

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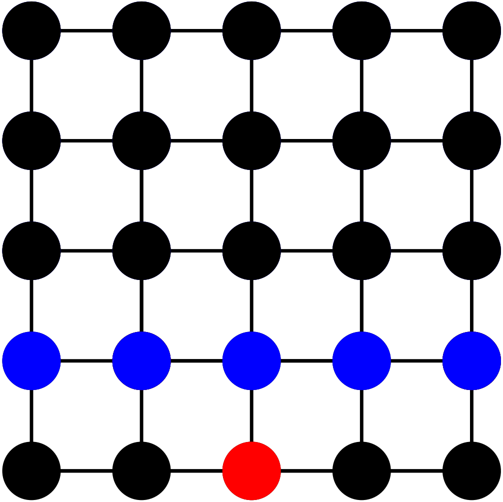

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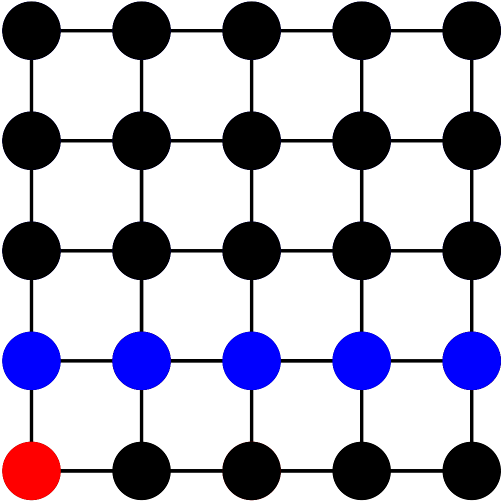

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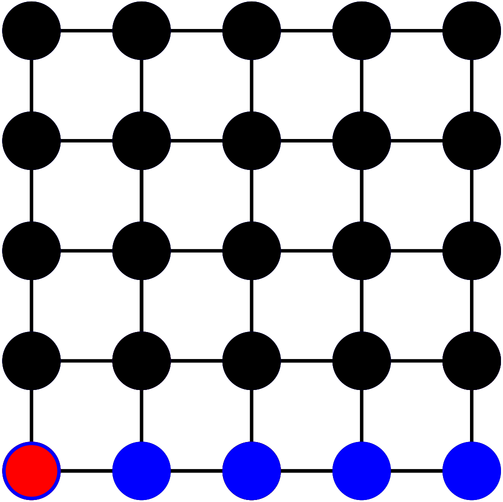

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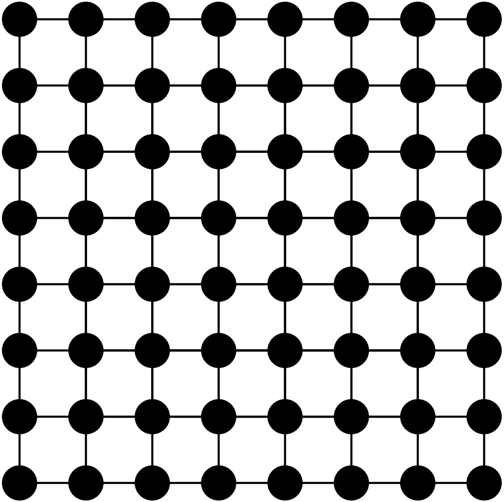
We need to explain how the robber can evade $n - 2$ cops.

The basic idea:

- Find a "safe" vertex with very few cops nearby.
- After the cops move, there still can't be many cops nearby, so there's a bit of room to maneuver.
- Robber works his way to an empty row; from there, he can get almost anywhere.
- Find another "safe" vertex, move there, and repeat.

How can the robber evade $n-2$ cops?

The interesting case: for some $k \in \{2, \ldots, n-2\}$, either the first k rows or last k rows contain at most $k-2$ cops. (Below, $n = 8$, $k = 5$, $k - 2 = 3$)



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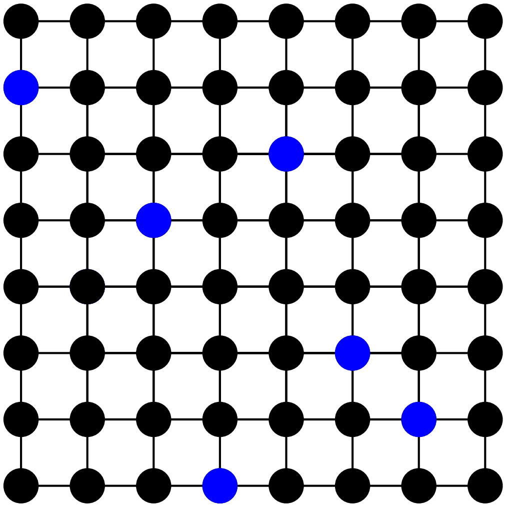

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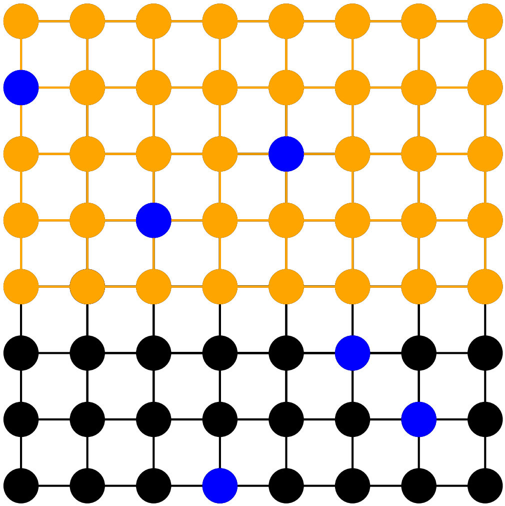

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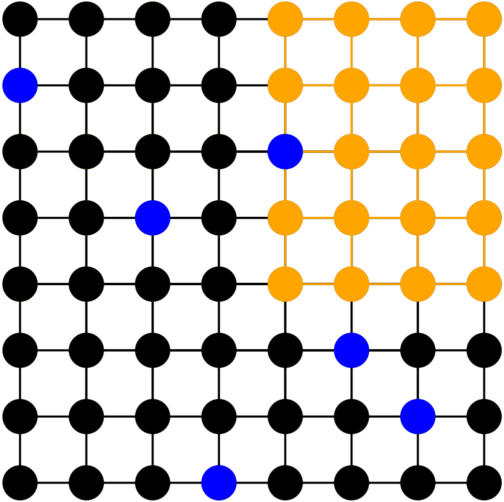

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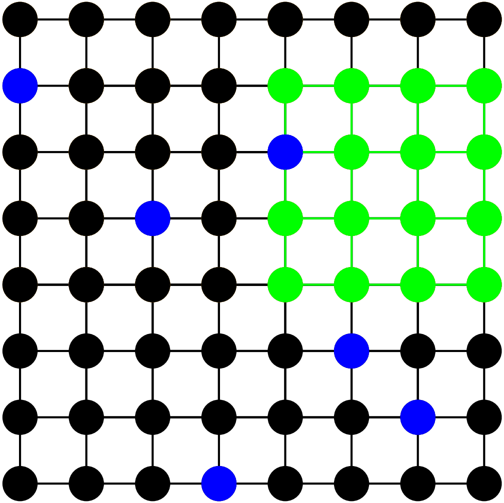

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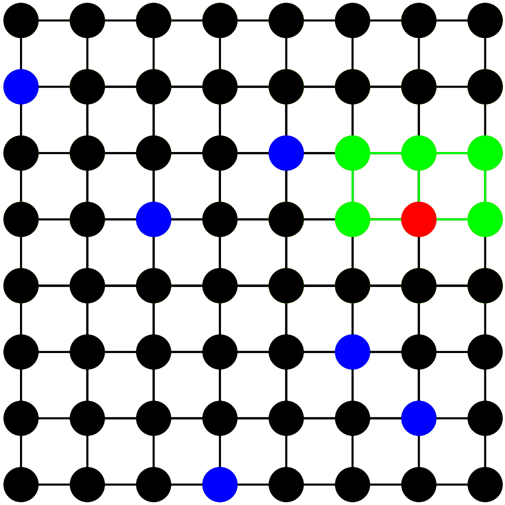

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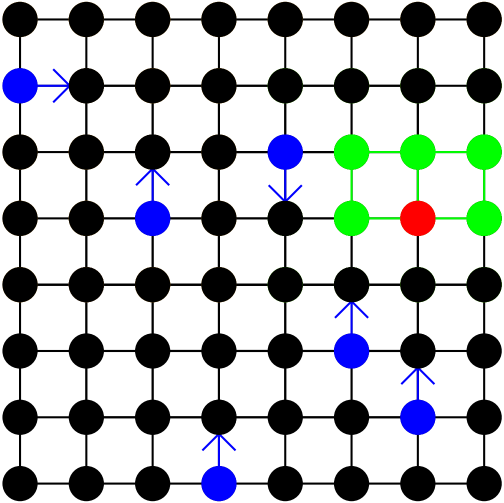

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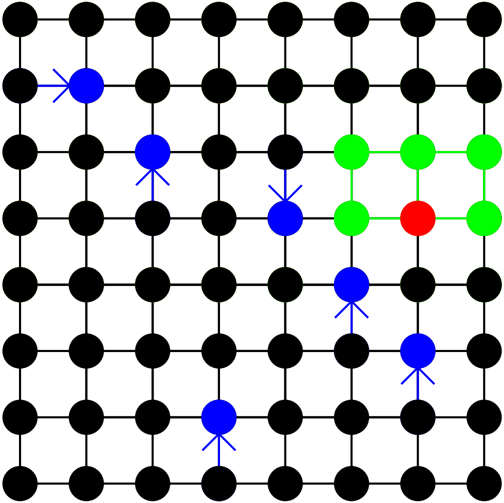

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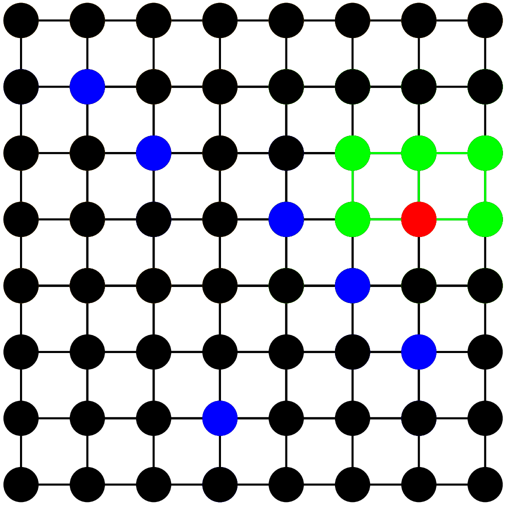

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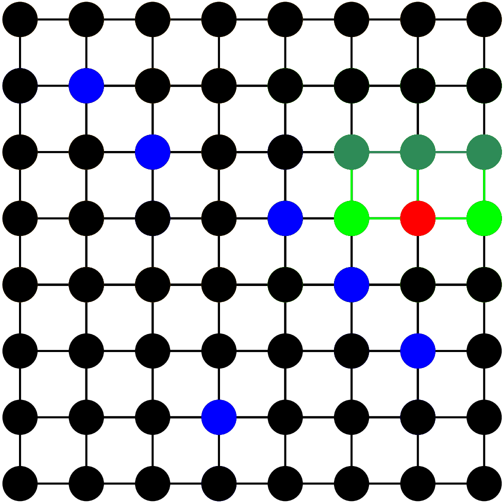

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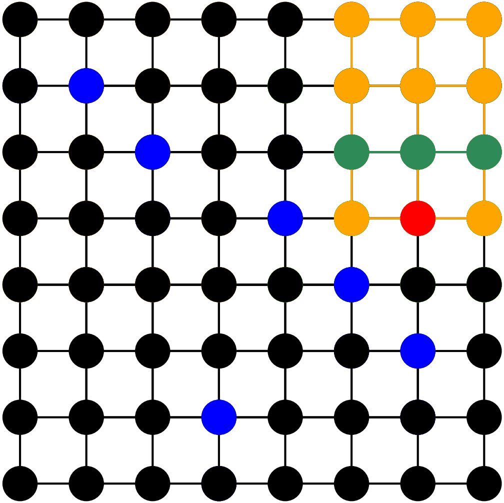

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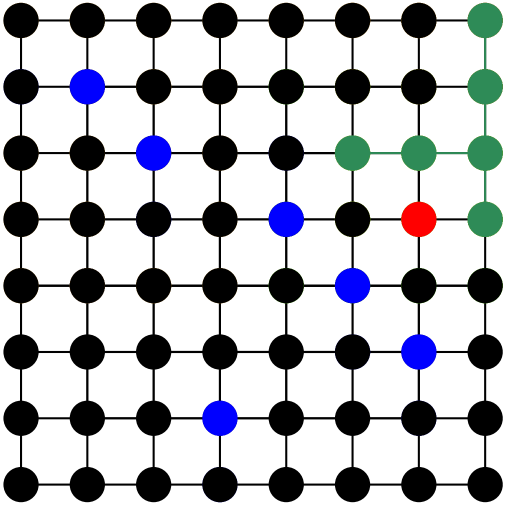

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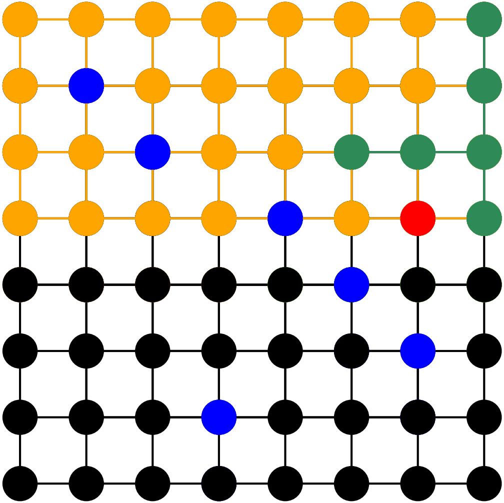

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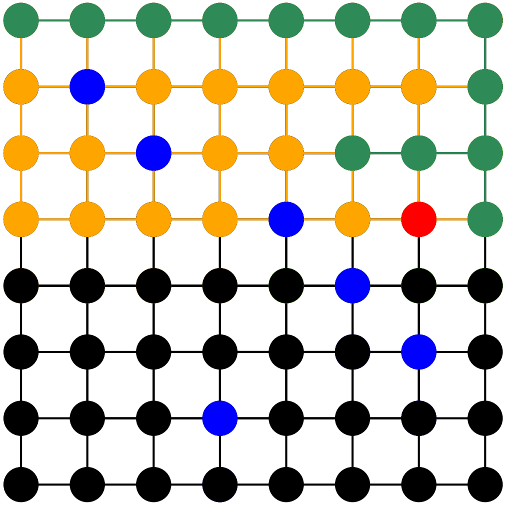

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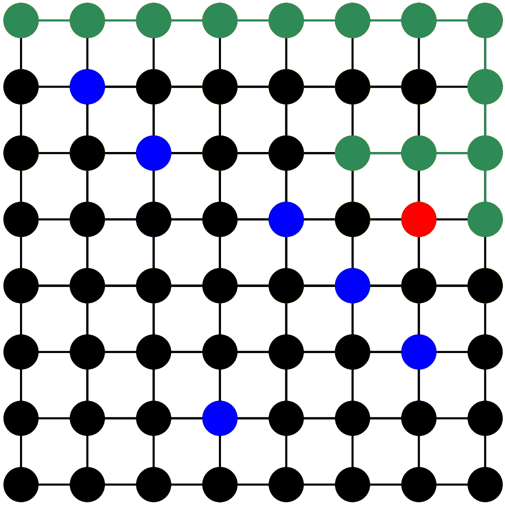

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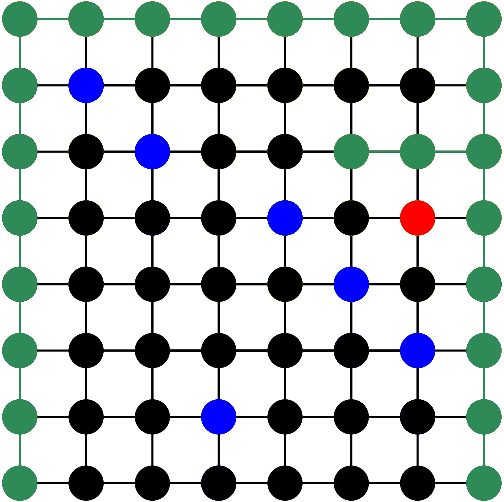

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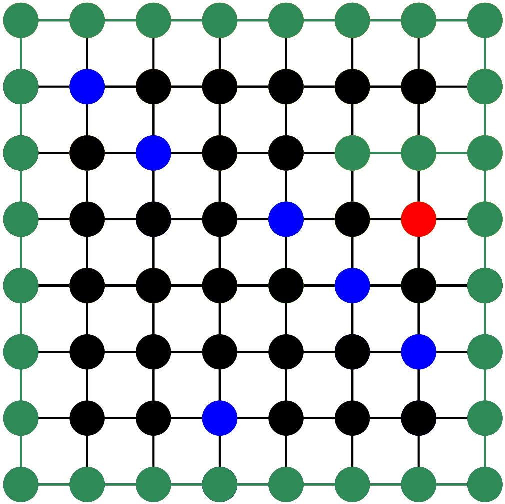

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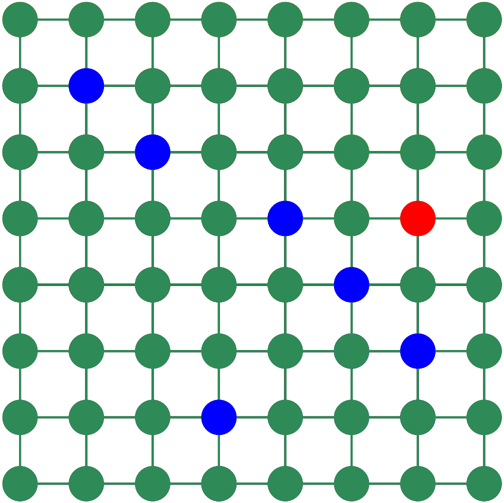

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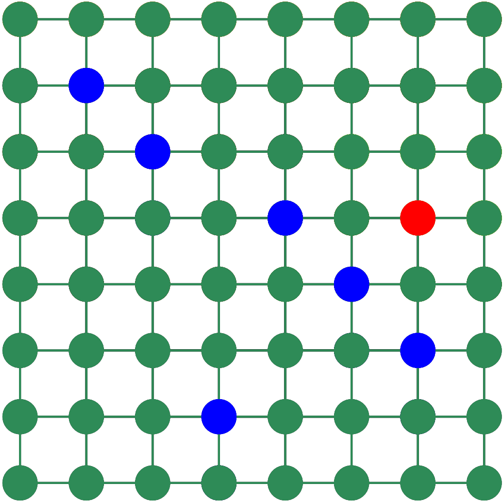

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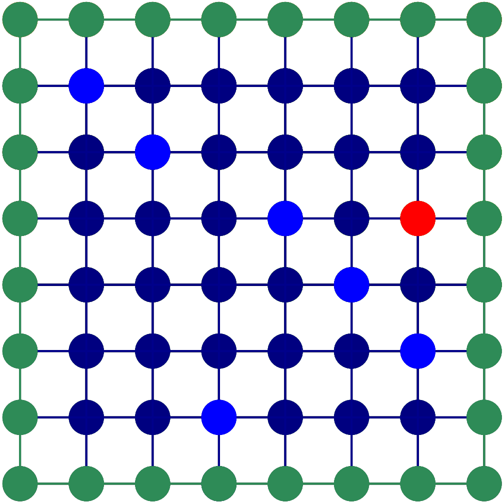


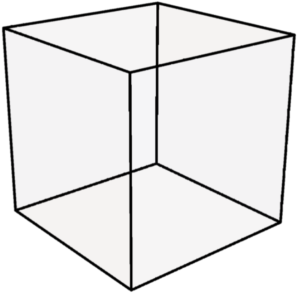
Three-dimensional grids

What if we add a third dimension? The obvious upper bound:

Proposition

For all n, we have $c_\infty(P_n \square P_n \square P_n) \leq n^2$.

Proof. Cops occupy the top level and move down.



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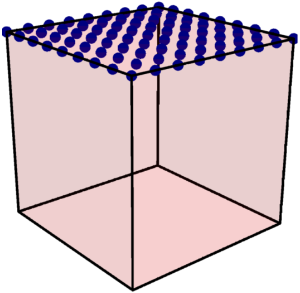

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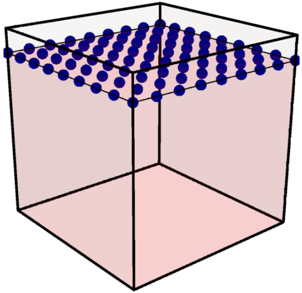

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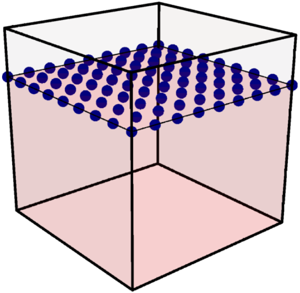

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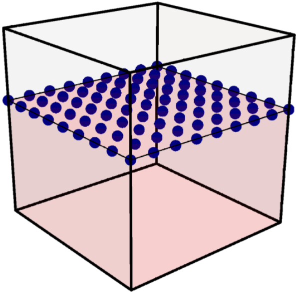

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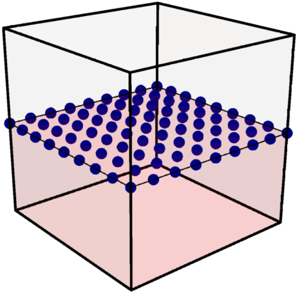

Three-dimensional grids

What if we add a third dimension? The obvious upper bound:

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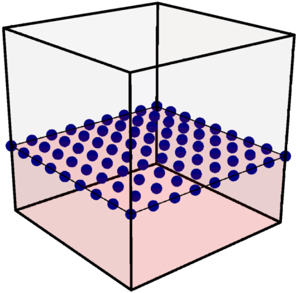

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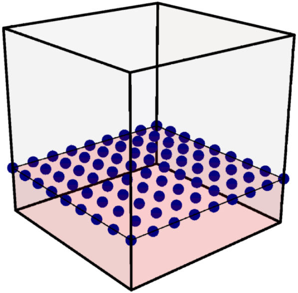

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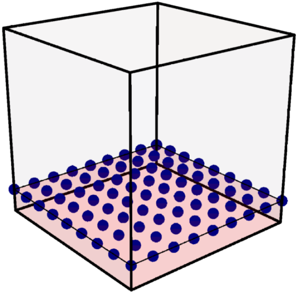

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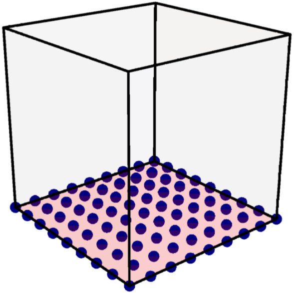

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Theorem (Kinnersley, Townsend '21+)

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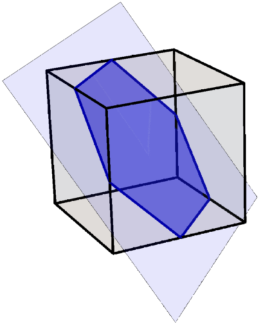

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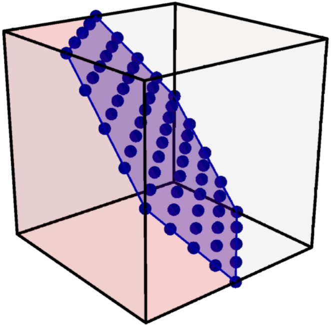

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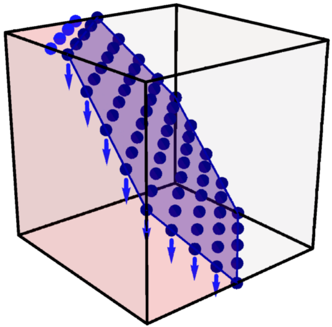

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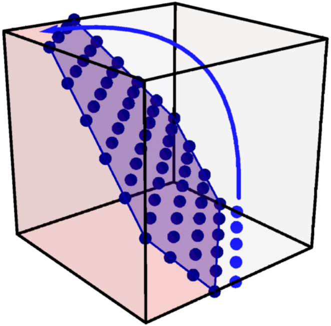

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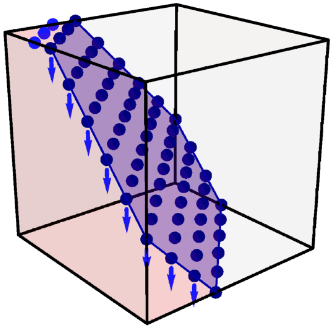

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For the lower bound, things are tougher.

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For sufficiently large n, we have $c_\infty(P_n \square P_n \square P_n) \geq 0.717n^2$.

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- In upper bound strategy, cops occupy cut set that cuts the graph in half, contains the robber, and shrink the robber's territory.
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- If C is the set of all vertices that the cops occupy, then $G - C$ can have multiple components, *but if* C *is small enough*, $G - C$ has a unique "largest" component.
- Robber needs two things:
    - A locally safe vertex where there aren't many cops, so that he has room to maneuver after the cops move.
    - To make sure he is in the large component after the cops move.

Minimizing Large Component

- If there are $\frac{3}{4}n^2$ cops, they can cut graph in half.
- With fewer cops, they can't, but they can come close! Cops want to make large component *as small as possible*.
- Bollabás, Leader '91 characterized the sets C which minimize the size of the large component of $G - C$. This show how the cops can play efficiently.

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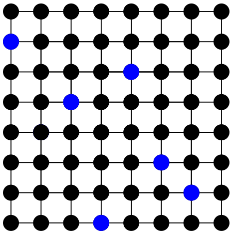

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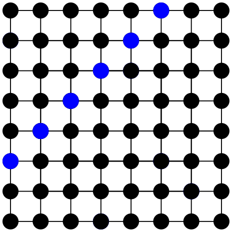

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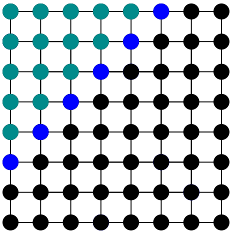
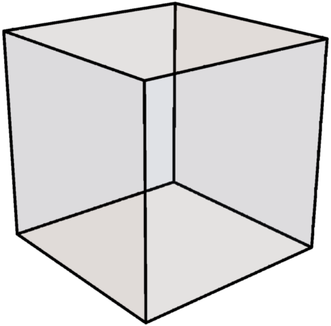


Finding a safe vertex

Theorem (Kinnersley, Townsend '21+)

For sufficiently large n, we have $c_\infty(P_n \square P_n \square P_n) \geq 0.717n^2$.

Proof idea. The robber looks for the half of the grid, H, with the fewest cops, then the quadrant, Q, then the octant, O. He finds a "safe" spot in O.



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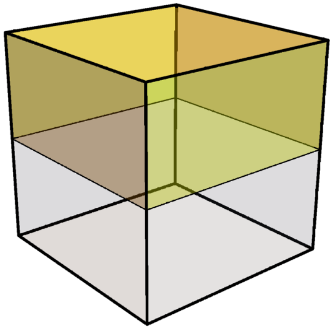

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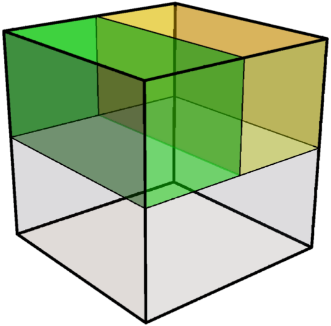

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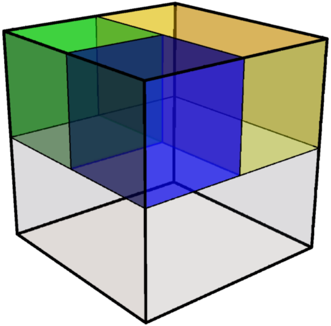
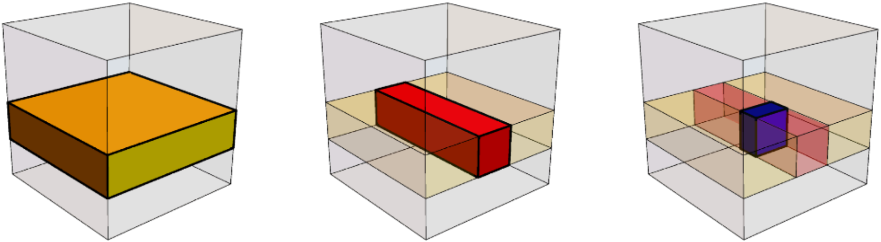


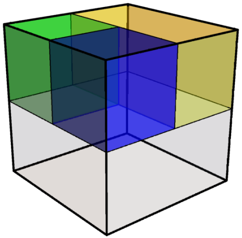
Finding a safe vertex, continued

Zooming in to the octant O, the robber finds a band of five "planes" with the fewest cops, then a $n/2 \times 5 \times 5$ subgrid (containing at most 8 cops), and finally a $3 \times 5 \times 5$ subgrid which will contain no cops.

Placing himself in the "middle" of this subgrid ensures that the robber will be able to access an empty row in the octant on his next turn.



Small and large components: evading $0.717n^2$ cops



After cops move, robber moves:

- Large component of O > small component(s) of Q
  → robber has access to large component of Q.
- Large component of Q > small component(s) of H
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- Bottleneck: Large component of H > small component(s) of G
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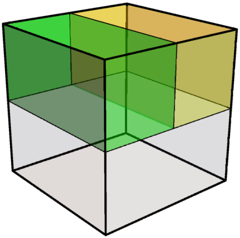

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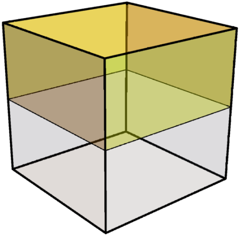

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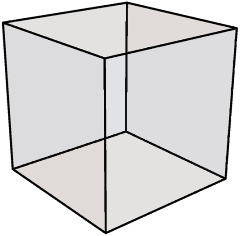

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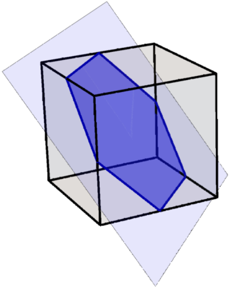
Higher-dimensional grids

Theorem (Kinnersley, Townsend '21+)

For the d-dimensional grid $P_n \square P_n \square \dots \square P_n$, *we have*

$$c_\infty(P_n \square P_n \square \dots \square P_n) \leq \sum_{k=0}^{\lfloor d/2 \rfloor} (-1)^k \binom{d}{k} \binom{\lfloor (d/2-k)n - d/2 - 1 \rfloor}{d-1}.$$



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# dimensions	Upper bound
4	$0.6667\, n^3$
5	$0.5990\, n^4$
6	$0.5500\, n^5$
10	$0.4305\, n^9$
100	$0.1380\, n^{99}$
1000	$0.0437\, n^{999}$

(Mehrabian showed $\frac{n^{d-1}}{d^2} \leq c_\infty(P_n \square \ldots \square P_n) \leq n^{d-1}$; it looks like our cop strategy gives $c_\infty(P_n \square \ldots \square P_n) \leq k \cdot \frac{n^{d-1}}{\sqrt{d}}$ for some k.)

Other Results

Theorem (Kinnersley, Townsend '21+)

For the discrete torus $C_n \square C_n$*, we have* $2n - 24 \leq c_\infty(C_n \square C_n) \leq 2n$.

(The proof is similar to the $P_n \square P_n$ proof.)

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The hypercube Q_n is the n-fold Cartesian product of P_2.

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For some constants k_1 *and* k_2*, we have* $k_1 \frac{2^n}{n \ln n} \leq c_\infty(Q_n) \leq k_2 \frac{2^n}{n}$.

Proof Idea. Upper bound: cops occupy a dominating set.

Lower bound: robber uses a potential function to avoid getting "cut off" by the cops.

Hypercube - lower bound

$$c_\infty(Q_n) \geq k_1 \frac{2^n}{n \ln n} \text{ for some constant } k_1.$$

Proof Idea.

- Robber wants to avoid getting "cut off" by cops.
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- There are $\binom{n}{d}$ vertices at distance d from a given vertex.

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- If potential is less than 1, robber is safe (for now), and cops have not surrounded him $\rightarrow$ Robber can access most of the cube.

Thanks!

- Thanks to the organizers of this conference.
- Thanks to my advisor, Dr. Kinnersley, for working with me on this project.
- Thank you all!

Questions?

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