Generalized Cops and Robbers: *N*-Player Pursuit Games on Graphs

Ath. Kehagias

(and G. Konstantinidis and M. Koutsmanis)

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1. Classic *Cops and Robbers* (CR) and its many variants.
2. Bonato and MacGillivray's 2-player *Generalized Cops and Robbers* games.
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They are special cases of

1. Stochastic games.
2. Recursive games.
3. Graphical games.
4. Reachability games.

Examples

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Example: Cyclic Pursuit

P_1, P_2, P_3 take turns moving along the edges of G.

1. P_1 wants to capture P_2.
2. P_2 wants to capture P_3.
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Example: Selfish Cops, Adversarial Robber

P_1, P_2, P_3 take turns moving along the edges of G.

1. Pursuers P_1 and P_2 want to capture P_3.
2. The pursuer who captures P_3 receives higher reward than the "other" pursuer.
3. P_3 wants to capture P_1

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States

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1. *Nonterminal states*:

$$S = \left\{ \left(x^1, ..., x^N, i \right) : \left(x^1, ..., x^N \right) \in V^n \text{and } i \in [N] \right\}$$

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   1. S_{nc} are the *noncapture states*.
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4. State set is $\overline{S} = S_{nc} \cup S_c \cup \{\tau\}$.

Histories and Strategies

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Histories are state sequences (in a play of the game).

1. Histories of length k : $H_k = \{\mathbf{s} = s_0 s_1 ... s_k\}$;
2. Histories of finite length: $H_* = \cup_{k=1}^{\infty} H_k$;
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Strategies map histories to next moves.

1. *Deterministic strategy*: $\sigma^n : H_* \to V$ (assigns a move to each finite-length history)
2. *Positional* strategy $\sigma^n (s_0 s_1 ... s_t) = \sigma^n (s_t)$.
3. *Notation*: $\sigma^{-n} = \left(\sigma^j\right)_{j \in [N] \backslash \{n\}}$
   E.g., $\sigma = \left(\sigma^1, \sigma^2, \sigma^3\right)$, $\sigma^{-1} = \left(\sigma^2, \sigma^3\right)$.

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1. *Total payoff* of the n-th player $Q^n : H_\infty \to \mathbb{R}$

$$Q^n(s_0, s_1, s_2, ...) = Q^n(s_0, \sigma) = \sum_{t=0}^{\infty} \gamma^t q^n(s_t)$$

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Each player tries to maximize his payoff.

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$$\left(G,\overline{S},S_c,\mathbf{q},\gamma\right)=\left\{\left(G,\overline{S},S_c,\mathbf{q},\gamma,s_0\right):s_0\in S\right\}$$

I.e., the same game played from all possible initial positions.

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When the context is clear, we will also write

$$\Gamma(G, s_0) \text{ and } \Gamma(G).$$

Example: Two-player (and two-token) Cops and Robbers

The capture set is

$$S_c = \{s : (x, x, p), x \in V, p \in [2]\}$$

The payoffs are

$$q^1(s) = -q^2(s) = \begin{cases} 1 & \text{iff} \quad s \in S_c, \\ 0 & \text{else;} \end{cases}$$

This is a zero-sum game.

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Then we have

$$Q^1(s_0, \sigma^1, \sigma^2) = -Q^2(s_0, \sigma^1, \sigma^2) = \gamma^{T_c(s_0, \sigma^1, \sigma^2)}$$

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P_1 wants to max (P_2 wants to min) $T_c(s_0, \sigma^1, \sigma^2)$.

Actually the equilibrium (*value*) of the game is

$$\sup_{\sigma^2} \inf_{\sigma^1} Q^1(s_0, \sigma^1, \sigma^2) = \inf_{\sigma^1} \sup_{\sigma^2} Q^1(s_0, \sigma^1, \sigma^2).$$

Example: Three-player Linear Pursuit

The capture set is $S_c = \widetilde{S}^1 \cup \widetilde{S}^2 \cup \widetilde{S}^{12}$

$$\begin{aligned} \widetilde{S}^1 &= \{s : (x,x,y,p), x \neq y \in V, p \in [3]\} \\ \widetilde{S}^2 &= \{s : (y,x,x,p), x \neq y \in V, p \in [3]\} \\ \widetilde{S}^{12} &= \{s : (x,x,x,p), x \in V, p \in [3]\} \end{aligned}$$

The payoffs are

$$q^1(s) = \begin{cases} 1 & \text{iff} \quad s \in \widetilde{S}^1 \cup \widetilde{S}^{12}, \\ 0 & \text{else}; \end{cases}$$

$$q^2(s) = \begin{cases} -1 & \text{iff} \quad s \in \widetilde{S}^1, \\ 1 & \text{iff} \quad s \in \widetilde{S}^2, \\ 0 & \text{else}; \end{cases}$$

$$q^3(s) = \begin{cases} -1 & \text{iff} \quad s \in \widetilde{S}^2 \cup \widetilde{S}^{12}, \\ 0 & \text{else}. \end{cases} \quad .$$

Example: Three-player Cyclic Pursuit

The capture set is $S_c = \widetilde{S}^1 \cup \widetilde{S}^2 \cup \widetilde{S}^3 \cup \widetilde{S}^{123}$ where

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\widetilde{S}^1 &= \{s : (x, x, y, p), x \neq y \in V, p \in [3]\} \\
\widetilde{S}^2 &= \{s : (y, x, x, p), x \neq y \in V, p \in [3]\} \\
\widetilde{S}^3 &= \{s : (x, y, x, p), x \neq y \in V, p \in [3]\} \\
\widetilde{S}^{123} &= \{s : (x, x, x, p), x \in V, p \in [3]\}
\end{aligned}$$

$$q^1(s) = \begin{cases} -1 & \text{iff} \quad s \in \widetilde{S}^3, \\ 1 & \text{iff} \quad s \in \widetilde{S}^1, \\ 0 & \text{else}; \end{cases}$$

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Example: Two Selfish Cops, One Adversarial Robber

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The payoffs are, with $\varepsilon \in \left[0, \frac{1}{2}\right]$,

$$\text{for } n \in \{1, 2\} : q^n(s) = \begin{cases} 1 - \varepsilon & \text{if } s \in \widetilde{S}^n, \\ \varepsilon & \text{if } s \in \widetilde{S}^m \text{ with } n \neq m, \\ \frac{1}{2} & \text{if } s \in \widetilde{S}^{12}, \\ 0 & \text{else.} \end{cases}$$

$$\text{for } n \in \{3\} : q^n(s) = \begin{cases} -1 & \text{if } s \in S_c, \\ 0 & \text{else.} \end{cases}$$

Positional Nash Equilibrium

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Theorem

For every GCR family $\Gamma(G)$ *there exists a positional deterministic strategy profile* $\widehat{\sigma} = \left(\widehat{\sigma}^1, ..., \widehat{\sigma}^N\right)$ *such that*

$$\forall s \in S, \forall n \in [N], \forall \sigma^n : Q_s^n(s, \widehat{\sigma}^n, \widehat{\sigma}^{-n}) \geq Q_s^n(s, \sigma^n, \widehat{\sigma}^{-n})$$

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   P_n may be able to achieve a payoff higher than $Q^n(s,\widehat{\sigma})$ if more than one players deviate from the strategy profile $\widehat{\sigma}$.

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2. This does not imply global optimality: P_n may be able to achieve a payoff higher than $Q^n(s,\widehat{\sigma})$ if more than one players deviate from the strategy profile $\widehat{\sigma}$.
3. In general, more than one NE will exist, some positional and some not.

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For every GCR family $\Gamma(G)$ there exists a **non***positional deterministic strategy profile $\widehat{\sigma} = \left(\widehat{\pi}^1, ..., \widehat{\pi}^N\right)$ such that*

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This is proved using *threat strategies*.

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Lemma

For each s and n, the game $\Gamma_n(G,s)$ has a value and the players have deterministic positional optimal strategies.

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Remark

The value and the optimal strategies can be computed by the *Value Iteration algorithm*.

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In the game $\Gamma_n(G,s)$, for every n,m, let

1. ρ_n^n be the optimal strategy of P_n.
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In the GCR game $\Gamma(G,s)$, the threat strategy $\widehat{\pi}^n$ of player n is defined as follows:

1. as long as every player $m \neq n$ follows ρ_m^m, player n follows ρ_n^n;
2. as soon as some player $m \neq n$ deviates from ρ_m^m, player n switches to ρ_m^n and uses it for the rest of the game.

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```
for $(x^1, x^2, p) \in S$ do
    if $(x^1, x^2, p) \in S_c$ then $U(x^1, x^2, p) = 1$
    else $U(x^1, x^2, p) = 0$
end
while $1 > 0$ do
    $U_{new} = U$
    for $(x^1, x^2) \in V^2$ do
        if $U(x^1, x^2, 1) \neq 0$ then
            $U_{new}(x^1, x^2, 1) = \min_{z \in N[x^1]} \gamma U(z, x^2, 2)$
        end
    end
    for $(x^1, x^2) \in V^2$ do
        if $U(x^1, x^2, 2) \neq 0$ then
            $U_{new}(x^1, x^2, 2) = \max_{z \in N[x^2]} \gamma U(x^1, z, 1)$
        end
    end
    if $U == U_{new}$ then break
    else $U = U_{new}$
end
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1. This algorithm computes the *value* (essentially the minimax capture time) of the CR game and can be easily extended to compute optimal strategies.
2. The algorithm has been presented by Hahn and MacGillivray ...also by Berarducci and Intrigila, Bonato and MacGillivray, Gavenčiak and others.
3. It is a special (deterministic) case of the stochastic games **Value Iteration** algorithm.

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2. The algorithm has been presented by Hahn and MacGillivray ...also by Berarducci and Intrigila, Bonato and MacGillivray, Gavenčiak and others.
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5. A detailed analysis of the algorithm appears in a 2021 paper by Kehagias and Konstantinidis

The Multi-Value Iteration Algorithm

Question

Can Value Iteration be applied to N-player games?

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What would *Multi*-value Iteration look like?

The Multi-Value Iteration Algorithm

For the case $N = 3$, let us look at the main part of the algorithm.

The Multi-Value Iteration Algorithm

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```
while $1 > 0$ do
    $U_{new} = U$
    for $\left(x^1, x^2, x^3, p\right) \in S$ do
        if $\left(x^1, x^2, x^3, p\right) \in S_{nc}$ then
            for $i \in [3]$ do
                $\widehat{u} = \arg\max_{u \in N[x^i]} \gamma U\left(u, x^{-i}, \mathbf{suc}(p), i\right)$
                $U_{new}\left(x^1, x^2, x^3, p, 1\right) = \gamma U\left(\widehat{u}, x^{-i}, \mathbf{suc}(p), 1\right)$
                $U_{new}\left(x^1, x^2, x^3, p, 2\right) = \gamma U\left(\widehat{u}, x^{-i}, \mathbf{suc}(p), 2\right)$
                $U_{new}\left(x^1, x^2, x^3, p, 3\right) = \gamma U\left(\widehat{u}, x^{-i}, \mathbf{suc}(p), 3\right)$
            end
        end
    end
    if $U == U_{new}$ then break
    else $U = U_{new}$
end
```

The Multi-Value Iteration Algorithm

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2. But when applied to trees and cycles, and with a little help (randomization) it *does* always terminate.
3. Termination is good enough because of the following.

Theorem

For every GCR game, if MVI terminates then the output $\widehat{\sigma}$ *is a deterministic positional NE of the game, i.e.:*

$$\forall n : \forall \sigma^n : \forall s_0 : Q^n\left(s_0, \widehat{\sigma}^n, \widehat{\sigma}^{-n}\right) \geq Q^n\left(s_0, \sigma^n, \widehat{\sigma}^{-n}\right).$$

The Multi-Value Iteration Algorithm

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Conjecture

For certain graph families (e.g., trees) there exists a single function from which the update operator is a contraction.

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For certain graph families (e.g., trees) there exists a single function from which the update operator is a contraction.

It cannot be so simple ...

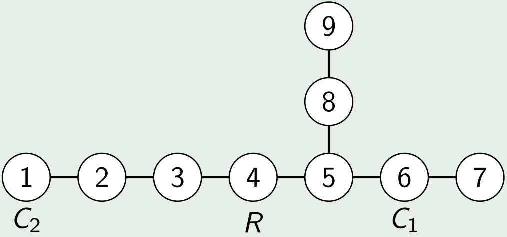
The Multi-Value Iteration Algorithm

Properties of SCAR: an Example

Example

Here minimizing capture time does not yield a NE. Specifically, a Cop may have an incentive to delay capture.

Initial positions as indicated, C_1 has the first move, $\varepsilon < \frac{1}{2}$.



Properties of SCAR: an Example

If the cops use CR-optimal strategies (i.e., go for the robber as fast as possible) the game evolves as follows:

Turn	0	1	2	3	4	5
C_1 **pos.**	6	5	5	5	4	4
C_2 **pos.**	1	1	2	2	2	3
R **pos.**	4	4	4	3	3	3

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The payoffs are

$$Q^1\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)=\varepsilon\gamma^5,$$
$$Q^2\left(s_0,\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)=(1-\varepsilon)\,\gamma^5,$$
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Now suppose C_2, R use their previous strategies, but C_1 first retreats to 7 and then moves directly towards the robber.

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Now suppose C_2, R use their previous strategies, but C_1 first retreats to 7 and then moves directly towards the robber.

Then R can move towards vertex 9 which, increases capture time and results in capture by C_1.

Properties of SCAR: an Example

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Properties of SCAR: an Example

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Turn	0	1	2	3	4	5	6	7	8	9	10	11	12	13
C_1 **pos.**	6	7	7	7	6	6	6	5	5	5	8	8	8	9
C_2 **pos.**	1	1	2	2	2	3	3	3	4	4	4	5	5	5
R **pos.**	4	4	4	5	5	5	8	8	8	9	9	9	9	9

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It is easy to see that

$$
\gamma^8 > \frac{\varepsilon}{1-\varepsilon} \Rightarrow Q^1\left(s_0, \widetilde{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right) > Q^1\left(s_0, \widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right).
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Hence $\left(\widehat{\sigma}^1, \widehat{\sigma}^2, \widehat{\sigma}^3\right)$ is not a NE.

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It is easy to see that

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Hence $\left(\widehat{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)$ is not a NE.
In fact, it can be proved that $\left(\widetilde{\sigma}^1,\widehat{\sigma}^2,\widehat{\sigma}^3\right)$ is a NE.

Properties of 3-Player SCAR

Properties of 3-Player SCAR

In what follows

$$\Omega^3 = \left\{ (\gamma, \varepsilon) : \gamma \in (0,1), \varepsilon \in \left[0, \frac{1}{2} \right] \right\}$$

and

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Theorem

For any G with $c(G) = 1$ the following holds:

$$\forall (\gamma, \varepsilon) \in \Omega^3, \forall s_0 \in S : \textit{every NE is capturing.}$$

Properties of 3-Player SCAR

Properties of 3-Player SCAR

Theorem

For any G with $c(G) = 2$ the following holds:

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Properties of 3-Player SCAR

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Theorem

For any G with $c(G) = 2$, let $\widehat{\sigma}$ be an optimal strategy profile in the two-cop CR game. Then

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For any G with $c(G) \geq 2$, the following holds:

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Properties of 3-Player SCAR

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For any G with $c(G) \geq 3$ the following holds:

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Properties of 3-Player SCAR

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Corollary

Given a graph G:

1. *suppose that for all $(\gamma, \varepsilon) \in \Omega^3$ and $s_0 \in S$, every NE is capturing; then $c(G) = 1$.*
2. *suppose that for all $(\gamma, \varepsilon) \in \Omega^3$ there exists some $s_0 \in S$ such that every NE is non-capturing; then $c(G) \geq 3$.*

Properties of 3-Player SCAR

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Corollary

G is cop-win iff : for all $(\gamma, \varepsilon) \in \Omega^3$ and $s_0 \in S$, every NE is capturing.

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For any G with $c(G) = 1$ the following holds:

$$\forall (\gamma, \varepsilon) \in \Omega^N, \forall s_0 \in S : \textit{every NE is capturing.}$$

Properties of N-Player SCAR

Properties of N-Player SCAR

Theorem

For any G with $c(G) \leq N-1$ the following holds:

$$\forall (\gamma, \varepsilon) \in \Omega^N, \forall s_0 \in S : \textit{there exists a capturing NE.}$$

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For any G with $c(G) \leq N-1$ the following holds:

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Properties of N-Player SCAR

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For any G with $c(G) \geq 2$, the following holds:

$\forall (\gamma, \varepsilon) \in \Omega^N, \exists s_0 \in S$: *there exists a non-capturing NE.*

Properties of N-Player SCAR

Theorem

For any G with $c(G) \geq 2$, the following holds:

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Theorem

For any G with $c(G) \geq N$ the following holds:

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Properties of N-Player SCAR

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Given a graph G*:*

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Corollary

G is cop-win iff: for all $(\gamma, \varepsilon) \in \Omega^N$ and $s_0 \in S$, every NE is capturing.

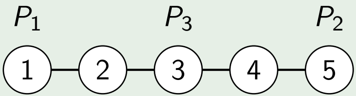
Properties of N-Player SCAR

Example of 3-Player Linear Pursuit

Example

The strategy profile $\overline{\sigma} = \left(\overline{\sigma}^1, \overline{\sigma}^2, \overline{\sigma}^3\right)$ defined as follows is a noncapturing NE.

1. $\overline{\sigma}^1$: P_1 stays in place as long as P_2 does not move; if P_2 moves, P_1 chases him.
2. $\overline{\sigma}^2$: P_2 stays in place as long as P_3 does not move; if P_3 moves, P_2 chases him.
3. $\overline{\sigma}^3$: P_3 stays in place as long as nobody moves; if P_1 moves, P_3 goes towards P_2; if P_2 moves, P_3 goes towards P_1.



Properties of 3-Player Linear Pursuit

Properties of 3-Player Linear Pursuit

Theorem

For every G:

$$\left(\forall s_0 \text{ there exists a capturing NE }\right) \Rightarrow c\left(G\right)=1.$$

Properties of 3-Player Linear Pursuit

Theorem

For every G:

$$(\forall s_0 \textit{ there exists a capturing NE }) \Rightarrow c(G) = 1.$$

Theorem

For every G:

$$c(G) > 1 \Rightarrow (\exists s_0 \textit{ for which we have a noncapturing NE.})$$

Properties of 3-Player Linear Pursuit

Properties of 3-Player Linear Pursuit

Theorem

If $c(G) = 1$ *then every optimal profile* $\widehat{\sigma}$ *of* $\Gamma_2(G, s_0)$ *is a NE of* $\Gamma(G, s_0)$.

Properties of 3-Player Linear Pursuit

Theorem

If $c(G) = 1$ *then every optimal profile* $\widehat{\sigma}$ *of* $\Gamma_2(G, s_0)$ *is a NE of* $\Gamma(G, s_0)$.

Theorem

If G *is a tree then every optimal profile* $\widehat{\sigma}$ *of* $\Gamma_2(G, s_0)$ *is a capturing NE in* $\Gamma(G, s_0)$.

Properties of 3-Player Linear Pursuit

Theorem

If $c(G) = 1$ *then every optimal profile* $\widehat{\sigma}$ *of* $\Gamma_2(G, s_0)$ *is a NE of* $\Gamma(G, s_0)$.

Theorem

If G *is a tree then every optimal profile* $\widehat{\sigma}$ *of* $\Gamma_2(G, s_0)$ *is a capturing NE in* $\Gamma(G, s_0)$.

Corollary

If G *is a tree, then*

$$\forall s_0 \text{ there exists a capturing NE.}$$

Properties of N-player Linear Pursuit

Properties of N-player Linear Pursuit

Theorem

For every G and every $N \geq 3$

$$(\forall s_0 \text{ there exists a capturing NE }) \Rightarrow c(G) = 1.$$

Properties of N-player Linear Pursuit

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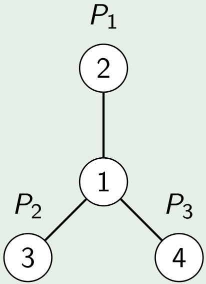
Theorem

For every G and every $N \geq 3$

$$c(G) > 1 \Rightarrow (\exists s_0 \textit{ for which we have a noncapturing NE})$$

Properties of Cyclic Pursuit

Example



Even though the graph is cop-win, the game has only non-capturing NE.

Thank You for your Attention!

Thank You!

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Thank You!

Questions?

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