THE MONOID AND GROUP OF COPYING GRAPHS

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Abstract. Recently proposed models of self-organizing networks like the web graph often incorporate some form of vertex copying in their design. The infinite limits of graphs generated by these models are almost surely isomorphic to the copying graphs, which are characterized by graph foldings and local versions of adjacency properties satisfied by the infinite random graph. Each finite connected graph H gives rise to an infinite copying graph R_H.

We study the endomorphisms and automorphisms of copying graphs. We prove that the natural order on the retracts of copying graphs embed all countable orders, while the endomorphism monoid of R_H is simple and embeds all countable semigroups. We consider which isomorphisms between finite induced subgraphs of R_H extend to automorphisms, and prove that all countable groups embed in the automorphism groups of copying graphs. The isomorphism types of the graphs R_H are related to the folding order on finite graphs. We study combinatorial properties of the folding order and prove that it embeds all finite orders.

1. Introduction

The web graph has vertices representing web pages, and edges representing the links between pages. It is a real-world self-organizing network possessing several billion vertices and edges, with vertices and edges appearing and disappearing over time. Another example of a self-organizing network is the protein-protein interaction network in a living cell. For recent surveys on the web graph and self-organizing networks, see [2, 4]. In self-organizing networks, each vertex acts as an independent agent, which will base its decision on how to link to the existing network on local knowledge. For this reason, many of the stochastic models of self-organizing networks incorporate some type of vertex copying in their design; see [12, 16]. For instance, consider the

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duplication model of [12]. In this model, vertices are added over a countable sequence of discrete time-steps, and p is a fixed probability in $(0, 1)$. At time $t+1$, an existing vertex u is chosen uniformly at random. A new vertex v is then added, and for each neighbour w of u, v is joined to w independently with probability p. Hence, the neighbours of the new node v form an imperfect copy of the neighbours of u.

In [6], a study was made of infinite limits of graphs generated by models of self-organizing networks. With probability 1, such limits satisfy the locally e.c. adjacency property, which is motivated by the local copying behaviour found in models of self-organizing networks. Define the *neighbourhood of* y *in* G, written $N_G(y)$, as the set $\{x : xy \in E(G)\}$. We will write $N(y)$ if G is clear from context. A graph G is *locally existentially closed* or *locally e.c.* if for all vertices y, finite $S \subseteq N(y)$, finite $T \subseteq V(G)\backslash S$, there is a vertex not in $S \cup T \cup \{y\}$ joined to each vertex of S and to no vertex of T. If we remove reference to y, then we have the e.c. property. The unique isomorphism type of countably infinite e.c. graph is named the *infinite random graph*, or the *Rado* graph, and is usually written R; see the surveys [9, 10]. In contrast to this property of R, [6] showed that there are $2^{\aleph_0}$ many non-isomorphic countable locally e.c. graphs.

Let H be a fixed finite connected graph, and let $R_0 \cong H$. Assume that R_t is defined and is countable. For each vertex $y \in V(R_t)$, and each non-empty subset $S \subseteq N(y)$, add a new vertex $x_{y,S}$ joined only to S. This gives the finite graph R_{t+1} which contains R_t as an induced subgraph. Define $R_H = \lim_{t\to\infty} R_t$, where $V(R_H) = \bigcup_{t\in\mathbb{N}} V(R_t)$ and $E(R_H) = \bigcup_{t\in\mathbb{N}} E(R_t)$. We refer to the graphs R_H as *copying graphs*. Copying graphs were first studied in [6] in the context of limits of web graph models. In [6] it was proved that R_H is locally e.c. but not e.c., and the chromatic number of R_H equals the chromatic number of H. It is not hard to see that R_H is connected with infinite diameter. (If H is the disjoint union of connected graphs X and Y, then R_H is isomorphic to the disjoint union of R_X and R_Y. For this reason, we will always assume H is connected in the definition of R_H.)

All graphs we consider are simple, undirected, and countable. We write $G \leq H$ if G is isomorphic to an induced subgraph of H. If $S \subseteq V(G)$, then we write $G \restriction S$ for the subgraph induced by S. If G and H are graphs, then a *vertex-mapping* from G to H is a function $f : V(G) \to V(H)$. We will abuse notation and write $f : G \to H$. We denote the image of f as $Im(f)$ or $f(G)$ (considered as a subset of $V(H)$). A vertex mapping $f : G \to H$ is an *embedding* if $G \restriction f(G) \leq H$.

A vertex mapping $f : G \to H$ with the property that $xy \in E(G)$ implies that $f(x)f(y) \in E(H)$ is a *homomorphism.* We write $G \to H$ to denote that G admits a homomorphism to H without reference to a specific mapping. A homomorphism from G to itself is an *endomorphism.* For additional background on graph homomorphisms, the reader is directed to [13]. The *endomorphism monoid* (or *monoid*) of G under composition is written $\mathrm{End}(G)$. The *automorphism group* (or *group*) of G is written $\mathrm{Aut}(G)$. We write $\aleph_0$ for the cardinality of the natural numbers $\mathbb{N}$ (which includes 0), and $2^{\aleph_0}$ for the cardinality of the real numbers.

The monoid of R was investigated in [3, 5, 11], while the group of R was studied by numerous authors; see the surveys [9, 10]. In the present article, we focus on the monoid and group of copying graphs, which we will demonstrate share some of the properties of the monoid and group of R. In Theorem 7, we prove that for each finite connected H, $\mathrm{End}(R_H)$ contains uncountably many principal ideals. From this, we derive that the natural order on retracts of R_H embeds each countable order. We prove in Theorem 10 that $\mathrm{End}(R_H)$ embeds all countable monoids; an analogous result for groups with regard to $\mathrm{Aut}(R_H)$ is proven in Theorem 11. Our techniques rely on certain retracts called *foldings endomorphisms* (see Section 2). Foldings of finite graphs give rise to an order, named the *folding order.* We investigate combinatorial properties of the folding order in Section 4, and prove in Theorem 16 that all finite orders embed in the folding order.

2. Endomorphisms of copying graphs

Graph homomorphisms are used in describing the induced subgraphs of R_H. The following theorem was proved in [7].

Theorem 1. *If G and H are countable graphs, then $G \leq R_H$ if and only if $G \to H$.*

A *retract* of G is an endomorphism f with the property that $f^2 = f$. In particular, f is the identity on its image. The set of all retracts of G is written $\mathrm{EEnd}(G)$. Although $\mathrm{EEnd}(G)$ is usually not a monoid, the retracts of G are equipped with an order: if $f, g \in \mathrm{EEnd}(G)$, then $f \leq g$ if and only if $fg = gf = f$. (We use the term "order" rather than "partial order".) We refer to this as the *natural order* on $\mathrm{EEnd}(G)$. For more information on the natural order of retracts, see [1].

Suppose that H, J are finite graphs such that $H \leq J$. Define $H \preccurlyeq_1 J$ if $H = J$ or there is a vertex v in J and u in $H = J - v$ such that u and v are non-joined, and $N(v) \subseteq N(u)$. We say that v *folds onto* u. We write $H \preccurlyeq J$ if there is a nonnegative integer m and graphs

$H_0 \cong H$, $H_1, \ldots, H_m = J$ so that $H_t \preccurlyeq_1 H_{t+1}$ for all $0 \leq t \leq m-1$. Note that the relation $\preccurlyeq$ is an order relation on the class of all finite graphs. We write this order as $(\mathcal{G}, \preceq)$, and name it the *folding order*. For example, $K_2 \preccurlyeq T$ where T is a tree, while cliques of different orders are incomparable in $(\mathcal{G}, \preceq)$. The graphs above K_2 in the folding order are sometimes called *cop-win* or *dismantlable graphs*; see [8].

We extend the folding to countable graphs as follows. Let H and J be countable graphs. The relation $H \preccurlyeq_1 J$ is defined exactly as in the finite case. We write $H \preccurlyeq J$ if there exists a countable sequence of order type at most ω (that is, finite, or the order type of $\mathbb{N}$) of countable graphs $(H_t : t \in \mathbb{N})$ so that

(1) $H_0 = H$,
(2) $H_t \preccurlyeq_1 H_{t+1}$ for all $t \in \mathbb{N}$, and
(3) $J = \lim_{t\to\infty} H_t$.

Without loss of generality, we may replace item (2) by: $H_t \preccurlyeq H_{t+1}$ for all $t \in \mathbb{N}$. For example, $K_2 \preceq G$, where G is an infinite one-way path (or *ray*). Note that for all $t > 0$, $H \preceq R_t$, and $R_t \preceq R_{t+1}$. Hence, $H \preceq R_H$. By Theorem 1, $C_6 \leq R_{K_2}$, despite the fact that $K_2 \not\preceq C_6$.

Following [8], we say that a finite graph H is *stiff* if it is minimal in $(\mathcal{G}, \preceq)$; that is, there is no $J \leq H$ such that $J \preceq H$. Cliques and cycles are stiff, while the only stiff finite tree is K_2. By Theorem 4.4 of [8], each graph H contains a unique (up to isomorphism) stiff induced subgraph, written $c(H)$, such that $c(H) \preceq H$. We refer to $c(H)$ as the *stiff-core* of H. The following theorem was proved in [7], and shows the close connection between the folding order and the graphs R_H.

Theorem 2. *Let H and J be finite connected graphs. If $H \preccurlyeq J$, then $J \leq R_H$ and $R_H \cong R_J$. In particular, $R_H \cong R_{c(H)}$.*

We will investigate properties of the folding order in Section 4. Let $J = \lim_{t\to\infty} H_t$ be a limit of a countable chain $\mathcal{C} = (H_t : t \in \mathbb{N})$ of graphs. Define $\mathrm{age}_{J,\mathcal{C}} : V(J) \to \mathbb{N}$ by

$$\mathrm{age}_{J,\mathcal{C}}(x) = \begin{cases} t & \text{if } x \in V(H_t)\backslash V(H_{t-1}) \text{ where } t > 0 \\ 0 & \text{else.} \end{cases}$$

We will simply write $\mathrm{age}(x)$ if J and $\mathcal{C}$ are clear from context. We now consider certain special retracts that will be useful for our discussion of the monoid and group of R_H. A retract $f : G \to G$ is a *folding endomorphism* if $G \restriction f(G) \preceq G$, and the set of all folding endomorphisms of G is written $\mathrm{FEnd}(G)$. We establish the connection between foldings and folding endomorphisms in the following lemma.

Lemma 3. *If $H \preceq G$, then there is $f \in \mathrm{FEnd}(G)$ with $f(G) = V(H)$.*

Proof. Suppose that $(G_t : t \in \mathbb{N})$ is a countable sequence of graphs so that $G_0 = H, G_t \preccurlyeq_1 G_{t+1}$ for all $t \in \mathbb{N}$, and $G = \lim_{t\to\infty} G_t$. Define $f : G \to G$ by induction on t. Let $f \restriction G_0 = 1_{G_0}$. For a fixed $t \geq 0$, assume that f is defined on G_t. Suppose that $v \in V(G_{t+1})\backslash V(G_t)$ folds onto u in $V(G_t)$. Define $f(v) = f(u)$, which is well-defined by inductive hypothesis.

Note that $f(G) = V(G_0) = V(H)$. The proof of this fact follows by induction on t and the definition of f. Hence, $f^2 = f$ and f is a retract. As $H \preccurlyeq G$, it remains only to prove that f is an endomorphism. For this, fix $xy \in E(G)$, and let $m = \max(\text{age}(x), \text{age}(y))$. We use induction on m to show that $f(x)f(y) \in E(G)$. If $m = 0$, then $f(x)f(y) = xy$, and so this case follows immediately. Assume that $f(x)f(y) \in E(G)$ if $m = t \geq 0$. Suppose now that $m = t+1$. Without loss of generality, let $\text{age}(x) = t+1$. Then $\text{age}(y) \leq t$ and x is the unique element of the set $V(G_{t+1})\backslash V(G_t)$. By the definition of f, we have that $f(x)y \in E(G)$. Since $\max(\text{age}(f(x)), \text{age}(y)) \leq t$, by induction hypothesis, we have that $f(x)f(y) = f^2(x)f(y) \in E(G)$. □

We will need the following theorem from [7] which characterizes the copying graphs.

Theorem 4. *If G is a connected locally e.c. graph such that $H \preceq G$, then $G \cong R_H$.*

We may define R_H in the case when H is countably infinite in a similar fashion to the case for H finite. The main difference is that for H infinite, the vertex $x_{y,S}$ introduced at time $t > 0$ may be joined to an infinite set S of vertices with $age(S) < t$. If H is locally finite (that is, the degrees of all vertices in H are finite), then $x_{y,S}$ is joined to an S which must be finite. The following result relates foldings of locally finite graphs and folding endomorphisms of R_H.

Corollary 5.
(1) *If H is a connected graph such that $R_0 \cong H$, then there is $f \in \mathrm{FEnd}(R_H)$ with $f(R_H) = V(R_0)$.*
(2) *If $H \preceq G$ and G is connected and locally finite, then there is $\alpha \in \mathrm{FEnd}(R_H)$ with $R_H \restriction \alpha(R_H) \cong G$.*

Proof. Item (1) follows by Lemma 3, since $R_0 \preceq R_H$. For (2), we have that $H \preceq G \preceq R_G$. We prove that $H \preceq R_G$. As R_G is connected and locally e.c., the result will follow by item (1) and Theorem 4. The result is immediate if G is finite, so we assume that G is infinite.

Suppose that $C_0 = (H_t : t \in \mathbb{N})$ is a countable sequence of finite graphs so that $H_0 \cong H, H_t \preccurlyeq_1 H_{t+1}$ for all $t \in \mathbb{N}$, and $G = \lim_{t\to\infty} H_t$. Let $D = (G_t : t \in \mathbb{N})$ be a countable sequence of countable graphs so that $G_0 \cong G, G_t \preccurlyeq_1 G_{t+1}$ for all $t \in \mathbb{N}$, and let $R_G = \lim_{t\to\infty} G_t$. To prove that $H \preceq R_G$, we must construct a chain $C = (J_t : t \in \mathbb{N})$ with $J_0 \cong H$, $J_t \preceq J_{t+1}$ for all $t \in \mathbb{N}$, and $R_G = \lim_{t\to\infty} J_t$. The rough idea of the proof is to *mix* the chains C_0 and D to form C. We define the graphs J_t inductively.

If x_1 is the unique vertex of $V(G_1)\backslash V(G)$, then $N_{G_1}(x_1)$ is a subset $N_G(y)$, for some vertex y in G. As G is locally finite, $N_G(x_1)$ is finite and so is contained in H_{i_1} for some $i_1 \in \mathbb{N}$. For $s \in \mathbb{N}$ define

$$J_s^1 = \begin{cases} H_s & \text{if } s \leq i_1 \\ G_1 \restriction (V(H_s) \cup \{x_1\}) & \text{else.} \end{cases}$$

Let C_1 be the chain $(J_s^1 : s \in \mathbb{N})$. Note that $G_1 = \lim_{t\to\infty} J_s^1$, $J_s^1 \preceq J_{s+1}^1$ for all $s \in \mathbb{N}$, and $H \preceq G_1$.

Proceeding inductively, we perform a similar construction with the unique vertex x_{t+1} of $V(G_{t+1})\backslash V(G_t)$. Let $i_{t+1} = i_t + 1 + \mathrm{age}(N_G(x_{t+1}))$. Then the finite set $N_{G_{t+1}}(x_{t+1})$ is contained in $G_{i_{t+1}}$.

For $s \in \mathbb{N}$ define

$$J_s^{t+1} = \begin{cases} J_s^t & \text{if } s \leq i_{t+1} \\ G_{t+1} \restriction (V(J_s^t) \cup \{x_{t+1}\}) & \text{else.} \end{cases}$$

Let C_{t+1} be the chain $(J_s^{t+1} : s \in \mathbb{N})$. Note that $G_{t+1} = \lim_{t\to\infty} J_s^{t+1}$, $J_s^{t+1} \preceq J_{s+1}^{t+1}$ for all $s \in \mathbb{N}$, and $H \preceq G_{t+1}$. Further, $J_{i_t}^t \preceq J_{i_{t+1}}^{t+1}$, by the definition of i_{t+1}.

Define $C = (J_t : t \in \mathbb{N})$ by

$$J_t = \begin{cases} J_{i_t}^t & \text{if } t > 0 \\ H & \text{else.} \end{cases}$$

By construction, $J_t \preceq J_{t+1}$ for all $t \in \mathbb{N}$, and $R_G = \lim_{t\to\infty} J_t$. □

Corollary 5 (2) implies that $\mathrm{FEnd}(R_H)$ is infinite (since there are infinitely many non-isomorphic locally finite connected G so that $H \preceq G$). This is in contrast to the case when H is a stiff-core: $\mathrm{FEnd}(H)$ has only a single element.

An order P *embeds* in an order Q if P is isomorphic to an induced suborder of Q. Bonato [3] proved that $(\mathrm{EEnd}(R), \leq)$ embeds all countable orders. We now prove that for all finite graphs H, $(\mathrm{EEnd}(R_H), \leq)$ is embeds all countable orders. For this, we consider the approach of [11], exploiting the ideals of $\mathrm{End}(R_H)$. An *ideal* of a semigroup S is a nonempty subset which is closed under multiplication on either side by elements of S. If $a \in S$, then the *principal ideal of S generated by a* is the smallest ideal (with respect to the inclusion order of ideals) which contains a. We begin with a simple but useful lemma. We use the notation J_e for the principal ideal of $\mathrm{End}(R_H)$ generated by the retract

e. For retracts e_1, e_2, we write $J_{e_1} \leq J_{e_2}$ if there are $f, g \in \mathrm{End}(R_H)$ such that $e_1 = fe_2g$. If $f : G \to H$ is a vertex mapping and $S \subseteq V(G)$, then we write $f \restriction S$ for the restriction of f to S.

Lemma 6. *Let $G_1, G_2 \leq R_H$ so that $H \preceq G_1$, G_2. For $i = 1, 2$, let $e_i : R_H \to G_i$ be a folding endomorphism. If $J_{e_1} \leq J_{e_2}$, then $G_1 \leq G_2$.*

Proof. By hypothesis, there are $f, g \in \mathrm{End}(R_H)$ such that $e_1 = fe_2g$. Then

$$fe_2g \restriction V(G_1) = e_1 \restriction V(G_1) = 1_{V(G_1)},$$

which implies that $e_2g \restriction V(G_1)$ is an injective homomorphism. If $u, v \in V(G_1)$ are not joined, then $e_2g(u)$ and $e_2g(v)$ are not joined. Hence, $e_2g \restriction V(G_1) : G_1 \to G_2$ is an embedding. □

We write $2^{\mathbb{N}}$ for the set of all subsets of the natural numbers. For a graph G, $\chi(G)$ denotes the chromatic number of G.

Theorem 7. *If H is a finite connected stiff graph, then there exists a set I of principal ideals of End(R_H) such that the inclusion order $(I, \subseteq)$ is isomorphic to $2^{\mathbb{N}}$.*

Proof. Fix r a vertex of H of maximum degree $\triangle$. As H is stiff, $\triangle > 1$. We will refer to r as the *root*. Define a graph G as follows. Join to r an infinite one-way ray, whose vertices are indexed by $\mathbb{N}$ (note that $r \neq n$, for all $n \in \mathbb{N}$). To each vertex $n \in \mathbb{N}$, join an endvertex n'.

Fix $X \subseteq \mathbb{N}$. Define G_X as the subgraph of G induced by the set

$$V(G) \backslash \{n' : n \notin X\}.$$

In other words, delete all vertices n' from G with n not in X. In particular, $G_{\mathbb{N}} = G$. Observe that G_X is locally finite for all choices of X.

Claim 1: For all $X \subseteq \mathbb{N}$, $H \preceq G_X \preceq G$, and $H \preceq G$.

We prove that $H \preceq G_X$; the proof that $G_X \preceq G$ is similar and so is omitted. Define $G_0 = H$. For $t \geq 1$, define G_t to be the subgraph of G_X induced by

$$V(H) \cup \{0, \dots, t-1\} \cup \{i' : i \in X, 0 \leq i \leq t-1\}.$$

Note that if $i \in X$, then i' folds to $i - 1$ (with $0'$ folding to r). Hence, for all $t \in \mathbb{N}$, $G_t \preceq G_{t+1}$. As $G_X = \lim_{t\to\infty} G_t$, we have that $G_X \preceq G$. That $H \preceq G$ by a similar argument.

It is straightforward to see that $G_X \to H$. Thus, G_X embeds in R_H by Theorem 1. As G is connected and locally finite, by Claim 1 and Corollary 5, there is a folding endomorphism $f \in \mathrm{FEnd}(R_H)$ with image G. By Claim 1 and Lemma 3, there is a folding endomorphism

f_X of G onto G_X. More explicitly, define the folding endomorphism $f_X : G \to G$ by

$$f_X(x) = \begin{cases} n-1 & x = n', n > 0, n \notin X \\ r & x = 0', 0 \notin X \\ x & \text{else.} \end{cases}$$

Claim 2: For $X, Y \subseteq \mathbb{N}$, $G_X \leq G_Y$ if and only if $X \subseteq Y$.

We prove only the forward direction, as the reverse direction is straightforward. Suppose that $g : G_X \to G_Y$ is an embedding. Let H_Z be the copy of H in G_Z, where Z is X or Y. We first show that $g(H_X) = H_Y$. Consider first the case when H is K_2 with vertices u and r. Then u is the unique vertex of H_X and H_Y that is degree 1 and joined to a vertex of degree 2. Hence, this case follows. Now consider the case when H has 3 or more vertices. If $g(H_X)$ contains some of the vertices n and n', then H contains some endvertices. Since $|V(H)| \geq 3$, these endvertices may be folded to vertices of H, which contradicts that H is a stiff.

Let r_X be the root of G_X and r_Y the root of G_Y. If H is K_2, then by the above discussion, $g(r_X) = r_Y$. Assume that $|V(H)| \geq 3$. Hence, $\deg_{G_X}(r_Z) = \triangle + 1$, and $\deg_{G_Z}(v) \leq \triangle$ for all v in $V(H_Z)$, where Z is X or Y. As $g(H_X) = H_Y$, we have that $g(r_X) = r_Y$. It is straightforward to see now that, by induction, g fixes vertices n and n' for all $n \in \mathbb{N}$. If $n \in X$ then $n' \in V(G_X)$, which implies that $g(n') = n' \in V(G_Y)$. Hence, $n \in Y$, and the proof of Claim 2 follows.

The proof of the theorem now follows if we prove that $J_{f_X f} \leq J_{f_Y f}$ if and only if $X \subseteq Y$. By Lemma 6 if $J_{f_X f} \leq J_{f_Y f}$, then $G_X \leq G_Y$. By Claim 2, $X \subseteq Y$.

Conversely, suppose that $X \subseteq Y$. We claim that $f_X = f_X f_Y$. If $u \in V(H)$, then $f_X f_Y(u) = u = f_X(u)$. Suppose $u = n$, or $u = n'$, where $n \in Y$. Then $f_X f_Y(u) = f_X(u)$. Now suppose that $u = n'$ with $n \notin Y$. If $n > 0$, then $f_X f_Y(n') = n - 1 = f_X(n')$. If $n = 0$, then $f_X f_Y(n') = r = f_X(n')$.

Note that $f f_Y = f_Y$ since $f_Y(V(G)) \subseteq V(G)$ and $f \restriction V(G) = 1_{V(G)}$. Therefore, $f_X f = f_X f_Y f = (f_X f)(f_Y f)$, which implies that $J_{f_X f} \leq J_{f_Y f}$. $\square$ $\square$

Theorem 7 supplies the following corollary.

Corollary 8. *Suppose that the graph H is finite, connected, and stiff. Then the following hold.*

(1) $\mathrm{End}(R_H)$ *is not simple.*
(2) *The natural order* $(\mathrm{EEnd}(R_H), \leq)$ *embeds all countable orders.*

Proof. Item (1) follows immediately from Theorem 7. For item (2), using the notation of the proof of Theorem 7, we observe that $f_X = f_X f_Y$. We need only show that $f_X = f_Y f_X$. To see this, fix $x \in V(G)$. As $X \subseteq Y$, if x is fixed by f_X, then x is fixed by f_Y. Hence, we need only consider the case when $x \notin X$. Suppose $x = n', n > 0$. Then

$$f_Y f_X(n') = f_Y(n-1) = n-1 = f_X(n').$$

If $x = 0'$, then

$$f_Y f_X(0') = f_Y(r) = r = f_X(0').$$

Hence, $X \subseteq Y$ if and only if $f_X \leq f_Y$ in $(\mathrm{EEnd}(R_H), \leq)$.

Therefore, the inclusion order $(2^{\mathbb{N}}, \subseteq)$ embeds in $(\mathrm{EEnd}(R_H), \leq)$. As $(2^{\mathbb{N}}, \subseteq)$ embeds all countable orders, the proof follows. $\square$

We now prove that the monoid $\mathrm{End}(R_H)$ is *universal*, in that it embeds all countable monoids. Given a finite graph H, form a graph H'_∞ as follows. Let the vertices of H be labelled $\{x_0 : x \in V(H)\}$. Let $V(H'_\infty) = \{x_i : i \in \mathbb{N}\}$ and $x \in V(H)\}$, where the x_i are all distinct vertices, and $x_i \neq y_j$ for distinct $x, y \in V(H)$. Define

$$E(H'_\infty) = \{x_i y_j : xy \in E(H)\}.$$

In other words, form H'_∞ by adding, for each vertex x, infinitely many vertices x_i which have the same neighbourhood in H'_∞ that x does (that is, H'_∞ is the lexicographic product of H with $\overline{K_{\aleph_0}}$). Although we required that H be finite in the definition of R_H, we may also consider H countably infinite. One difference in the construction is that at each time-step R_t, a countably infinite graph results. For the universality of $\mathrm{End}(R_H)$, we need the following result proven in [6].

Theorem 9. *For a finite graph H, $R_H \cong R_{H'_\infty}$.*

If X is a set, define the *transformation monoid on X*, written $T(X)$, to be the set of functions $f : X \to X$, with operation equalling composition. By an analogue of Cayley's theorem for monoids, each monoid of cardinality $|X|$ embeds in $T(X)$.

Theorem 10. *For a finite graph H, the monoid $T(X)$ embeds in $\mathrm{End}(R_H)$, where X is a countably infinite set. In particular, each countable monoid embeds in $\mathrm{End}(R_H)$.*

Proof. By Theorem 9, it is sufficient to prove the result for $H = H'_\infty$. We first observe that $T(X)$ embeds in $\mathrm{End}(H)$. To see this, fix $x = x_0$ in H, and identify X with $\{x_i : i \in \mathbb{N}\}$. Fix a mapping $g : X \to X$. Define $G : H \to H$ which acts as g on X, and fixes all other vertices. As the x_i have the same neighbours in R_H, it follows that G is an endomorphism of H. Define $\beta : T(X) \to \mathrm{End}(H)$ by $\beta(g) = G$. It is straightforward to check that β is an injective monoid homomorphism.

We next prove that there exists an injective monoid homomorphism $\alpha : \mathrm{End}(H) \to \mathrm{End}(R_H)$. Once this is established, then $\alpha\beta : T(X) \to \mathrm{End}(R_H)$ supplies an embedding of $T(X)$ into $\mathrm{End}(R_H)$.

Fix f an endomorphism of H, with H considered as the induced subgraph R_0 of R_H. Let $F_0 = f$. For $t \geq 0$, assume that F_t is an endomorphism of R_t, and $F_t \restriction R_0 = F_0$. Define F_{t+1} by

$$F_{t+1}(z) = \begin{cases} F_t(z) & \text{if } z \in V(R_t); \\ x_{F_t(y),F_t(S)} & \text{if } z = x_{y,S}, \text{ where } y \in V(R_t),\ S \subseteq N(y). \end{cases}$$

It is straightforward to check (using the definition of R_{t+1} and the fact that $F_t \in \mathrm{End}(R_t)$) that F_{t+1} is an endomorphism of R_{t+1}. Note that $F_{t+1} \restriction R_t = F_t$.

The map $F = \bigcup_{t\in\mathbb{N}} F_t$ is an endomorphism of R_H. Hence, the mapping $\alpha : \mathrm{End}(H) \to \mathrm{End}(R_H)$ defined by $\alpha(f) = F$ is well-defined. It is straightforward to see that α is injective, and that $\alpha(1_H) = 1_{R_H}$. Fix $f, g \in \mathrm{End}(H)$ and $z \in V(R_H)$. We prove by induction on the age t of z that

$$\alpha(fg)(z) = \alpha(f)\alpha(g)(z). \tag{1}$$

(1) will establish that α is an embedding of monoids, and is immediate if $t = 0$. Fix $t \geq 0$. Suppose that $age(z) = t+1$ and so z is of the form $x_{y,S}$, where $y \in V(R_t)$, $S \subseteq N(y) \subseteq V(R_t)$. Then

$$\begin{aligned} \alpha(fg)(z) &= x_{\alpha(fg)(y),\alpha(fg)(S)} \\ &= x_{\alpha(f)\alpha(g)(y),\alpha(f)\alpha(g)(S)} \\ &= \alpha(f)\alpha(g)(z). \end{aligned}$$

The second equality follows since the ages of y and S are strictly less than t, and by induction hypothesis. □

3. Automorphisms of copying graphs

Henson [14] proved that $\mathrm{Aut}(R)$ embeds (that is, contains subgroups isomorphic to) all countable groups. Our first result of this section proves a similar result for $\mathrm{Aut}(R_H)$. Given a set X, we use the notation $\mathrm{Sym}(X)$ for the group of permutations of X. We have the following.

Theorem 11. *The group* $\mathrm{Sym}(X)$ *embeds in* $\mathrm{Aut}(R_H)$*, where* X *is countably infinite. In particular, each countable group embeds in* $\mathrm{Aut}(R_H)$.

Proof. Define H'_∞ as in the proof of Theorem 10, and so $R_H \cong R_{H'_\infty}$ by Theorem 9. Therefore, by Cayley's theorem, it is sufficient to prove that $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(R_{H'_\infty})$.

To see this, observe first that $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(H'_\infty)$. The proof is similar to the one that $T(X)$ embeds in $\mathrm{End}(H'_\infty)$ given in the proof of Theorem 10. Using the technique of the proof of Theorem 10, it follows that $\mathrm{Aut}(H'_\infty)$ embeds in $\mathrm{Aut}(R_{H'_\infty})$. Hence, $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(H'_\infty)$, as desired. □

The graph R is *homogeneous*, in the sense that any isomorphism between finite induced subgraphs of R extends to an automorphism of R. Hence, R displays as much symmetry as possible. In particular, R is vertex- and arc-transitive. The countable homogeneous graphs were classified by Lachlan, Woodrow [15]. From that classification, we deduce that R_H is homogeneous if and only if H is K_1 (since $R_{K_1} \cong \overline{K_{\aleph_0}}$).

The graphs R_H are not in general vertex-transitive.

Lemma 12. *The graph R_{K_3} is not vertex-transitive.*

Proof. List the vertices of K_3 at time $t = 0$ as a, b, c. Consider the vertex $x = x_{a,\{b\}}$ added at $t = 1$. We claim that x is not in a K_3 in R_{K_3}. Hence, R_{K_3} is not vertex-transitive since there is no automorphism sending a to x. We proceed by induction on $t \geq 1$ to prove that x is not in a K_3 in R_t. This clearly holds for $t = 1$. Suppose it is true for a fixed $t \geq 1$, and consider R_{t+1}. For a contradiction, suppose that x is in a K_3 with vertices x, y, z. One of these vertices, say z, must have age $t + 1$. Let z' be the vertex z copied from at time $t + 1$. Note that $x, y \in N(z) \subseteq N(z')$. Therefore, $z' \neq x, y$. Then x, y, z' forms a K_3 in R_t, which is a contradiction. □

With this result in mind, a natural question (if somewhat vague) is how symmetric is R_H? More precisely, which pairs of induced subgraphs of R_H have the property that every isomorphism between them extends to an automorphism of R_H? We do not have a complete answer to this question for R_H, even if $H \cong K_2$. We do, however, have the following result.

Let G and G' be induced subgraphs of J. We say that G and G' are *simply reflective*, written $G \sim_1 G'$, if their vertex sets are of the form $Z \cup \{x\}$ and $Z \cup \{y\}$ respectively, where x and y are non-joined and their neighbour sets within Z are equal. Hence, $J \restriction (Z \cup \{x, y\})$ folds onto both G and G'. We say that G and G' are *reflective*, written $G \sim G'$ if $G = G'$, or there are induced subgraphs $G_0, G_1, \ldots, G_r$ of J with $G_0 = G$ and $G_r = G'$, such that $G_{i-1} \sim_1 G_i$ for all $i = 1, \ldots, r$.

Theorem 13. *Let H be a finite, connected, stiff graph, and let G and G' be finite induced subgraphs of H such that $H \preceq G, G'$ and $G \sim G'$ in R_H. Then any isomorphism between G and G' extends to an automorphism or R_H.*

Proof. We prove the theorem in the case when $G \sim_1 G'$; the proof when $G \sim G'$ follows by induction. Let $G = R_H \restriction (V(G) \cup V(G'))$. Then $H \preceq G$ and G is connected, and so by Theorem 2, $R_H \cong R_G$. Moreover, G has an automorphism carrying G to G', and this extends to an automorphism of R_G (using a similar technique as in the proof of Theorem 10). □

Corollary 14. *The graph R_{K_2} is arc-transitive.*

Proof. Consider two copies G and G' of K_2 in R_{K_2}. The existence of a path connecting G to G' witnesses that $G \sim G'$ in R_{K_2}. Since $K_2 \preceq G, G'$, the proof follows by Theorem 13. □

4. The folding order

We now investigate combinatorial properties of the folding order. Let $(\mathcal{G} \preceq)$ represent the folding order on the class of finite graphs. An order $(P, \leq)$ is *scattered* if it does not embed the order-type of the rational numbers. If $x \in P$, then we use the notation $\uparrow x$ for the set $\{y \in P : x \leq y\}$.

Theorem 15.
1. *The minimal elements of $(\mathcal{G}, \preceq)$ are the stiff graphs. If G and H are non-isomorphic stiff graphs, then $(\uparrow G) \cap (\uparrow H) = \emptyset$.*
2. *There is an order-homomorphism from $(\mathcal{G}, \preceq)$ into $(\mathbb{N}, \leq)$. In particular, $(\mathcal{G}, \preceq)$ is scattered.*
3. *The order $(\mathcal{G}, \preceq)$ has infinite height and width.*

Proof. The proof of the first statement of (1) follows from the definitions. Suppose that some $J \in \uparrow (G \cap \uparrow H)$. Then J folds to the stiff induced subgraphs G and H, which contradicts that stiff-cores are unique up to isomorphism.

For (2), first note that each element of $(\mathcal{G}, \preceq)$ has finite height. For $t \in \mathbb{N}$, define A_t to be the elements of $(\mathcal{G}, \preceq)$ which are of height t. Define $f : (\mathcal{G}, \preceq) \to (\mathbb{N}, \leq)$ by $f(A_t) = t$. It is straightforward to see that f is an order-homomorphism.

For (3), note that $K_2 \preceq P_3 \preceq P_4 \preceq \ldots$ implies that $(\mathcal{G}, \preceq)$ has infinite height. Since the infinite set of all stiff graphs forms an antichain, the order $(\mathcal{G}, \preceq)$ has infinite width. □

By Theorem 15 (2), $(\mathcal{G}, \preceq)$ does not embed every countable order. However, it has a relatively rich structure as demonstrated in the following theorem. We write $\deg_G(x)$ for the degree of a vertex x in G.

Theorem 16. *The folding order $(\mathcal{G}, \preceq)$ embeds all finite orders.*

Proof. We use the well-known fact that each finite order $(P, \leq)$ embeds in a hypercube $\mathbf{2}^n$, for some positive integer n. It is therefore sufficient to prove that for all positive n, $\mathbf{2}^n$ embeds in the folding order $(\mathcal{G}, \preceq)$.

Fix $n \geq 3$ a positive integer. Label the vertices of the complete graph K_n as $\{0, 1, \ldots, n-1\}$. Form the graph G by adjoining 2^i many disjoint endvertices to the vertex i of K_n, where $1 \leq i \leq n-1$. Note that the degree sequence of G consists of a constant sequence of 1's of length $2^n - 2$ followed by the sequence

$$(n-1, n+1, \ldots, n-1+2^{n-1}).$$

In particular, each vertex i in G that is not an endvertex has degree $n-1 \geq 2$ or higher, and any two vertices i, j from G have degrees differing by at least 2. In addition, $K_n \preceq G$ by folding all the endvertices joined to i onto j, where $i \neq j$.

We identify the hypercube $\mathbf{2}^n$ with the set of all binary n-sequences ordered coordinatewise. If $\sigma, \tau \in \mathbf{2}^n$, then we write $\sigma \leq \tau$ for this ordering. For a binary n-sequence σ, let $\sigma(i)$ be the ith position of σ, and let $s(\sigma) = \sum_{i=1}^{n} \sigma(i)$.

Define a graph G_σ containing G as an induced subgraph. The vertex i from G we label i_σ in G_σ. To each vertex i_σ, add $\sigma(i)$ many disjoint endvertices. Then $G \preceq G_\sigma$. The degree sequence of G_σ consists of $(2^n - 2) + s(\sigma)$ many 1's, followed by

$$\lambda = (n-1+\sigma(1), n+1+\sigma(2), \ldots, n-1+2^{n-1}+\sigma(n)).$$

Note that each two distinct entries of λ differ by 1 or more.

Let $\mathcal{G}(n)$ be the subordering of $(\mathcal{G}, \preceq)$ induced by the set $\{G_\sigma : \sigma \in 2^n\}$. Define $\varphi : \mathbf{2}^n \to \mathcal{G}(n)$ by $\varphi(\sigma) = G_\sigma$. We claim that φ is an order-isomorphism. The map φ is clearly surjective. It is injective, since distinct sequences in $\mathbf{2}^n$ give rise to graphs with distinct degree sequences in $\mathcal{G}(n)$.

Suppose that $\sigma \leq \tau$ in $\mathbf{2}^n$. We embed G_σ in G_τ by identifying each i_σ with i_τ, and mapping corresponding sets of endvertices. Hence, $G_\sigma \leq G_\tau$, and since endvertices joined to i_τ fold to j_τ with $i \neq j$, we have that $G_\sigma \preceq G_\tau$.

Now suppose that $G_\sigma \preceq G_\tau$. Hence, $G_\sigma \leq G_\tau$ via an embedding f. Since

$$\deg_{G_\sigma}(i_\sigma) = \begin{cases} n-1+2^i+\sigma(i) & \text{if } i > 0 \\ n-1+\sigma(0) & \text{else,} \end{cases}$$

it follows that $f(i_\sigma)$ is not an endvertex of G_τ. Thus, for all i, $f(i_\sigma) = j_\tau$ for some j. Define

$$\alpha : \{0, 1, \ldots n-1\} \to \{0, 1, \ldots n-1\}$$

by $\alpha(i) = j$. The mapping α is injective as f is an embedding, and so is a bijection. For all $x, y \in \{0, 1\}$ and all $0 \leq i \leq n-1$, we have that $n - 1 + x < n - 1 + 2^i + y$. Further, for all $x, y \in \{0, 1\}$, $n - 1 + 2^i + x < n - 1 + 2^j + y$ if and only if $i < j$. Therefore, for all $0 \leq i \leq n - 1$, $\alpha(i) \geq i$, and so α is the identity mapping on $\{0, 1, \ldots n - 1\}$. In particular, for all i, $f(i_\sigma) = i_\tau$. Suppose now that for some $i \in \{0, 1, \ldots n - 1\}, \tau(i) < \sigma(i)$. But then

$$\deg_{G_\tau}(f(i_\sigma)) = \deg_{G_\tau}(i_\tau) < \deg_{G_\sigma}(i_\sigma),$$

which is a contradiction. Hence, $\sigma \leq \tau$. □

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