Cops and Robbers on Graphs Based on Designs

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Abstract: We investigate the cop number of graphs based on combinatorial designs. Incidence graphs, point graphs, and block intersection graphs are studied, with an emphasis on finding families of graphs with large cop number. We generalize known results on Meyniel extremal families by supplying bounds on the incidence graphs of transversal designs, certain G-designs, and BIBDs with $\lambda \geq 1$. Families of graphs with diameter 2, C_4-free, and with unbounded chromatic number are described with the conjectured asymptotically maximum cop number.
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1. INTRODUCTION

Cops and Robbers is vertex-pursuit game played on graphs that has been the focus of much recent attention. We consider only finite simple graphs. There are two players consisting of a set of *cops* and a single *robber*. The game is played over a sequence of discrete time-steps or *rounds*, with the cops going first in the first round and then playing alternate time-steps. The cops and robber occupy vertices, and more than one cop may occupy a vertex. When a player is ready to move in a round they may move to a neighboring vertex, or *pass* by remaining on their own vertex. Observe that any subset of cops may move in a given round. The cops win if, after some finite number of rounds, one of them can occupy the same vertex as the robber. This is called a *capture*. The robber wins if he can avoid capture indefinitely. A *winning strategy for the cops* is a set of rules

that, if followed, result in a win for the cops. A *winning strategy for the robber* is defined analogously.

If we place a cop at each vertex, then the cops are guaranteed to win. Therefore, the minimum number of cops required to win in a graph G is a well-defined positive integer, named the *cop number* of the graph G. We write $c(G)$ for the cop number of a graph G. Nowakowski and Winkler [18], and independently Quilliot [20], considered the game with one cop only; the introduction of the cop number came in [2]. Many papers have now been written on cop number since these three early works; see the book [4] for additional references and background on the cop number.

Meyniel's conjecture is one of the deepest unsolved problems on the cop number. It states that for a connected graph G of order n, $c(G) = O(\sqrt{n})$. Hence, the largest cop number of a graph is asymptotically bounded by $d\sqrt{n}$ for a constant d. The conjecture has so far resisted all attempts to resolve it, and the best known bounds do not even prove that $c(G) = O(n^{\varepsilon})$, for $\varepsilon < 1$.

Until recently, the only graphs known to attain the conjectured maximum cop number were incidence graphs of projective planes. To be more precise, an infinite family of graphs $(G_n : n \geq 1)$ is *Meyniel extremal* if there is a constant d such that for all n, $c(G) \geq d\sqrt{n}$. Hence, assuming Meyniel's conjecture is true, Meyniel extremal families are those with the asymptotically maximum cop number. As proved in [2], if the *girth* of G (i.e., the length of a shortest cycle) is at least 5, then $c(G) \geq \delta(G)$, where $\delta(G)$ is the minimum degree of G. The incidence graph of the projective plane PG$(2, q)$ of order q a prime power has girth 6, is $(q + 1)$-regular, with $2(q^2 + q + 1)$-many vertices. Hence, these incidence graphs form a Meyniel extremal family. (For a nonprime power order, see the proof of Corollary 3.2 and Lemma 3.3.)

The present article is motivated by two goals. First, we describe new Meyniel extremal families of graphs. It is of interest that all such known families arise from combinatorial designs. Meyniel's conjecture was settled for graphs of diameter 2, but no explicit example was given of such a family; see [15]. We solve a problem stated in [4] by describing a Meyniel extremal family (the polarity graphs) whose members have diameter two and unbounded chromatic number; see Corollary 3.2. To bound the cop number of polarity graphs, we develop a new technique for a lower bound on the cop number based on forbidden subgraphs; see Lemma 2.1.

We summarize the results on new Meyniel extremal (or "ME") families in this paper in the chart below. All the graphs in the chart are incidence graphs (denoted by "IG") except for polarity graphs and t-orbit graphs. We list the order of the graphs G in the family, the degrees of their vertices, a lower bound for their cop number, and a reference to the appropriate theorem in the paper. Let q be a prime power, and let α, t, and u denote constants that do not depend on the other parameters (such as q).

A second motivation is to investigate the cop number of graphs based on designs in their own right. While there are many constructions of graphs based on designs, we focus on three types of such graphs: incidence graphs, block intersection graphs, and point graphs. In the case of incidence graphs, in Section 4 we consider the cop number of group-divisible designs (GDDs), transversal designs, and truncated transversal designs. Along the way, we give lower bounds for the cop number and exact values in some cases. We generalize some of our bounds to the general settings of BIBDs with $\lambda \geq 1$, G-designs, and t-designs for $t \geq 3$. We derive bounds on the cop number of block intersection graphs for BIBDs in Section 5. In the final section, we include a discussion of the cop number of point graphs of partial geometries.

ME family	Order of G	Degrees	$c(G)$	Reference
polarity graphs	$q^2 + q + 1$	$q, q+1$	$q/2$	Corollary 3.2
t-orbit graphs	$(q^2 - 1)/t$	$q, q+1$	$q/(t+1)$	Corollary 3.4
IG oval designs	$(3)2^{2i-1} - 2^{i-1} - 1$	$2^{i-1},$	2^{i-1}	Theorems 4.1,4.3
	for $i \geq 2$	$2^i + 1 - \frac{1}{2^{i-1}-1}$		
IG Denniston	$\beta + \frac{\beta(\beta-1)}{2^i(2^i-1)},$	$2^i, 2^{i+\alpha} + 1$	2^i	Theorems 4.1,4.3
designs	$\beta = 2^{2i+\alpha} + 2^i - 2^{i+\alpha}$			
IG TD$(q+1, q)$	$2q^2 + q$	$q+1, q$	q	Corollary 4.6
IG TD(k, q),	$2q^2 - \alpha q$	$q - \alpha, q$	$q - \alpha$	Corollary 4.6
$q = k + \alpha$				
IG TDD(k, q, u),	$2q^2 - \alpha q + u$	$k, k+1, q$	$\min\{k, q\}$	Corollary 4.8
$q = k + \alpha$				

We assume familiarity with design and graph theory; for a reference on designs see [9], and for a reference on graphs see [23]. All the graphs we consider are undirected, finite, and simple. We denote the minimum degree of a graph G by $\delta(G)$, and its chromatic number by $\chi(G)$. A graph is *regular* if every vertex has a fixed degree, and is (a, b)-*regular* if each vertex has degree a or b. The *dual* of a design is formed by switching the role of points and blocks. Let $PG(2, q)$ be the projective plane over the finite field of order a prime power q, which we denote by $GF(q)$.

2. TOOLS

We present a lemma, which while elementary to prove, gives rise to new families of graphs with the conjectured asymptotically largest value possible for the cop number. A graph G is *H-free* if it does not contain H as a subgraph. The complete bipartite graph with parts of cardinality m and n is denoted by $K_{m,n}$.

Lemma 2.1. *Let $t \geq 1$ be an integer. If G is $K_{2,t}$-free, then $c(G) \geq \delta(G)/t$.*

Proof. Suppose there are k cops playing, where $k < \delta(G)/t$. We show first that the robber may choose a vertex that is not adjacent to any cop in the first round. Let S be the set of vertices occupied by the cops, and let u be a vertex not in S. If no vertex of S is adjacent to u, then the robber can pick vertex u. Now assume that $N(u) \cap S \neq \emptyset$. Since G is $K_{2,t}$-free, a cop C that is adjacent to u can be adjacent to at most $t - 1$ other neighbors of u. Hence, C is equal or adjacent to at most t neighbors of u. Similarly, a cop not adjacent to u can be adjacent to at most $t - 1$ neighbors of u. It follows that the number of neighbors of u which are equal or adjacent to a cop is at most kt. But since $kt < \delta(G)$, this means that there must be some neighbor v of u, which is not adjacent to any cop, and the robber can begin the game on v.

Now assume that in round $r \geq 0$ of the game, the cops and robber have moved and the robber is on the vertex u so that no cop is adjacent to u. Suppose in round $r + 1$ the cop "attacks" the robber by moving to some vertex of his neighbor set. A similar argument to that in the initial round shows that there is a neighbor v of u which is not adjacent to any cop, so the robber can move to v in round $r + 1$ and avoid capture. $\square$

As a consequence of Lemma 2.1, we have the following.

Corollary 2.2. *If G is C_4-free, then $c(G) \geq \delta(G)/2$.*

In the case that G has neither a 3- nor 4-cycle, we recall the following result of Aigner and Fromme.

Lemma 2.3 ([2]). *If G is a connected graph with girth at least 5, then $c(G) \geq \delta(G)$.*

3. POLARITY GRAPHS

We now describe a family of Meyniel extremal graphs that are diameter 2. Fix q a prime power. The *Erdős-Rényi graphs*, written ER(q), have vertices the points of PG(2, q), and u is adjacent to v if $u^T v = 0$ (where we identify vertices with one-dimensional subspaces of GF(q)^3). These are well-known examples of graphs that are C_4-*free extremal*, in the sense that they have the largest possible number of edges in a C_4-free graph (which is asymptotic to $(1/2)n^{3/2}$); see [7, 12].

The Erdős-Rényi graphs are part of the more general family of polarity graphs. For a given PG(2, q) with points P and lines L, a *polarity* $\pi : P \to L$ is a bijection such that for all points p_1 and p_2, $p_1 \in \pi(p_2)$ if and only if $p_2 \in \pi(p_1)$. The *polarity graphs* are formed on vertex set P by joining distinct vertices u and v if $u \in \pi(v)$. For example, the orthogonal polarity (which exists for all PG(2, q)) gives rise to the Erdős-Rényi graphs. For more on polarity graphs, see [17].

Polarity graphs have order $q^2 + q + 1$, with $q(q + 1)^2$ edges, and each vertex has degree $q + 1$ or q. These graphs are C_4-free, have diameter 2, and possess unbounded chromatic number as $q \to \infty$ [14].

We have the following theorem that is crucial in proving that polarity graphs are Meyniel extremal. For a vertex u, we use the notation $N(u)$ for the neighbor set of u, $N[u]$ for the closed neighbor set of u, and $N_2(u)$ for vertices of distance 2 to u.

Theorem 3.1. *Suppose that G satisfies the following properties.*

1. *The graph G has order $q^2 + q + 1$ with each vertex of degree $q + 1$ or q.*
2. *The graph G is C_4-free and diameter 2.*

Then

$$q/2 \leq c(G) \leq q + 1.$$

Proof. The proof of the lower bound follows by Corollary 2.2. For the upper bound, we play with $q + 1$ cops. Without loss of generality, assume the robber is safe (that is, not adjacent to a cop) on the vertex u. Note that the graph induced by $N[u]$ may contain triangles, but the subgraph induced by $N(u)$ consists of vertices of either degree 0 or 1. Hence, $N[u]$ consists of triangles joined only at u, along with vertices of degree 1 joined at u. Now move one cop C to $N(u)$, which forces R to move off u. The cop C moves then to u, forcing the robber to move to $N_2(u)$; note that the robber cannot move back to $N(u)$. The remaining q cops move to q distinct vertices of $N(u)$. If there is a vertex w in $N(u)$ with no cop (in the case that u has degree $q + 1$), move C to w. Now every vertex of $N_2(u)$, including R, is adjacent to some cop. The robber then loses in the next round. □

We do not know the exact value of the cop number of polarity graphs. However, the bounds in Theorem 3.1 prove the following, which answers a question raised implicitly in [15], and explicitly in [4] (see p. 71). A vertex u is a *corner* if there is some vertex v distinct from u such that $N[u] \subseteq N[v]$.

Corollary 3.2.

1. *There exists a Meyniel extremal family whose graphs have diameter 2, and whose members have unbounded chromatic number.*
2. *There exists a Meyniel extremal family whose graphs are C_4-free, and whose members have unbounded chromatic number.*

Proof. For a positive integer n, let q be the least prime power such that $q^2 + q + 1 \leq n$. If $n = q^2 + q + 1$, then we are done by using the polarity graphs and Theorem 3.1. If $q^2 + q + 1 < n$, then we form graphs G_1 and G_2 from a polarity graph G with order $q^2 + q + 1$ as follows. The graphs G_i will constitute the Meyniel extremal families in item (i) in the statement of the corollary, for $i = 1, 2$.

Let $m = n - (q^2 + q + 1)$. For G_1, fix a vertex x and add m new vertices x_i (where $1 \leq i \leq m$) adjacent to vertices in $N[x]$ and to no other vertices. The vertices x_i are corners, and the addition of corners does not change the cop number; see [2] or Chapter 2 of [4]. Hence, $c(G_1) = c(G)$, and it is straightforward to see that the diameter of G_1 is 2. As $\chi(G_1) \geq \chi(G)$, we note that the graphs G_1 have unbounded chromatic number as $n \to \infty$ (as the same statement holds for G with $q \to \infty$).

For G_2, add m-many vertices y_i (where $1 \leq i \leq m$) of degree 1 to a fixed vertex x. As vertices y_i are corners, we have that $c(G_2) = c(G)$. It is easy to see that G_2 does not contain 4-cycles, and $\chi(G_2) = \chi(G)$.

The remainder of the proof now follows from the Bertrand–Chebyshev theorem [8], which states that for all integers $x > 1$, there is a prime q between x and $2x$. In particular, for $i = 1, 2$, let $x = \lfloor (1/2)\sqrt{n - \sqrt{n} - 1} \rfloor$. It follows that there is a prime q such that

$$(1/2)\sqrt{n - \sqrt{n} - 1} \leq q \leq \sqrt{n - \sqrt{n} - 1}.$$

Hence, $q \leq \sqrt{n}$ and so $q^2 + q + 1 \leq n$. Now $q/2 = (1/4)\sqrt{n}(1 - o(1))$. Therefore, for every fixed $\epsilon > 0$ and for sufficiently large n, we derive the bound that

$$c(G_i) \geq q/2 \geq (1/4 - \epsilon)\sqrt{n}. \qquad \square$$

We note that the method described in the proof of Corollary 3.2 gives a general technique to provide Meyniel extremal families via infinite families missing some orders. We summarize the method in the following lemma.

Lemma 3.3. *Let X be a set of positive integers with the property that for all integers $y \geq 1$, there is an $x \in X$ such that $y \leq x \leq 2y$. Suppose that for all $x \in X$ there exists a graph G_x with order $ax^2 + bx + d$ for rationals $a > 0$, b, d with either $a \geq 1$ or $b = 0$ (note that b and d can be negative), such that $c(G_x) \geq mx$, for some fixed rational $m \in (0, 1]$. Then for $n \geq 1$ an integer, there exists a graph G_n of order n with the property that for n sufficiently large,*

$$c(G_n) \geq \frac{m}{2}\sqrt{\frac{n}{a}}(1 - o(1)).$$

In particular, $(G_n : n \geq 1)$ forms a Meyniel extremal family.

Proof.   Assume first that $b \geq 0$. Let $y = \left\lfloor \frac{1}{2}\sqrt{\left(\frac{n}{a} - \frac{b}{a}\sqrt{n} - \frac{d}{a}\right)} \right\rfloor$ and choose $x \in X$ such that $y \leq x \leq 2y$ (we choose n so that y is positive). Then we have that for large enough n,

$$ax^2 + bx + d \leq a\left(\frac{n}{a} - \frac{b}{a}\sqrt{n} - \frac{d}{a}\right) + b\sqrt{\frac{n}{a} - \frac{b}{a}\sqrt{n} - \frac{d}{a}} + d$$

$$\leq n - b\sqrt{n} - d + b\sqrt{n} + d$$

$$= n,$$

where the second inequality follows since either $b\sqrt{\frac{n}{a} - \frac{b}{a}\sqrt{n} - \frac{d}{a}} \leq b\sqrt{\frac{n}{a}} \leq b\sqrt{n}$ in the case $a \geq 1$, or trivially if $b = 0$. Form G_n from the graph G_x by adding k-many vertices of degree 1 to a fixed vertex, where $k = n - (ax^2 + bx + d)$. We therefore have that

$$c(G_n) = c(G_x)$$

$$\geq mx$$

$$\geq m\left\lfloor \frac{1}{2}\sqrt{\frac{n}{a} - \frac{b}{a}\sqrt{n} - \frac{d}{a}} \right\rfloor$$

$$= \frac{m}{2}\sqrt{\frac{n}{a}}(1 - o(1)).$$

If $b < 0$, then write $b = -b'$, with $b' > 0$. Since $ax^2 + bx + d \leq ax^2 + b'x + d$, the proof now follows from the case $b > 0$ replacing b with b'. $\qquad \square$

We may apply Lemma 3.3 to polarity graphs with $a = b = d = 1$, $m = 1/2$, and X the set of all primes.

Examples described in [13] give $K_{2,t+1}$-free extremal graphs G with order $(q^2 - 1)/t$, where q is a prime power and $t \geq 1$ is an integer. In GF(q), fix h an element of order t, and let $H = \{1, h, \ldots, h^{t-1}\}$. The vertices of G are the t-element orbits of $(\text{GF}(q) \times \text{GF}(q))\backslash\{(0, 0)\}$ under the action of multiplication by powers of H. Two classes $\langle a, b \rangle$ and $\langle c, d \rangle$ are adjacent if $ac + bd \in H$. We call these the *t-orbit graphs*. Every vertex of G is degree q or $q + 1$. By Theorem 3.1, $c(G) \geq q/(t + 1)$. Hence, by Lemma 3.3 (with $a = 1/t$, $b = 0$, $d = -1/t$, $m = 1$, and X the set of all primes) we have the following result.

Corollary 3.4.   *Let $t \geq 1$ be an integer. There exists a Meyniel extremal family whose graphs are $K_{2,t+1}$-free.*

4. INCIDENCE GRAPHS

We consider in this section the cop number of incidence graphs of general classes of designs. The *incidence graph* of an incidence structure with point set X and line or

block set $\mathcal{B}$ is the bipartite graph with vertex set $X \cup \mathcal{B}$, such that $x \in X$ is adjacent to $B \in \mathcal{B}$ if and only if x lies in B. It is a part of folklore that Meyniel extremal families can be obtained from the incidence graphs of projective planes. It was shown in [3] that Meyniel extremal families may be constructed as the incidence graphs of the partial designs obtained by deleting a fixed number of parallel classes from an affine plane.

A *balanced incomplete block design* with positive integer parameters v, k, and λ (or BIBD (v, k, λ)) is an incidence structure with v points, whose blocks have size k, so that each pair of distinct points is in exactly λ blocks. The *replication number*, written r, is the number of blocks containing a given point (the parameter r is a constant independent of whichever point is chosen). It is evident that $k \leq v$, $bk = vr$, and $\lambda(v - 1) = r(k - 1)$. We begin with the following result dealing with the case $\lambda = 1$. The cases for higher λ will be discussed below in Corollary 4.9.

Theorem 4.1.   *If Γ is the incidence graph of a* BIBD$(v, k, 1)$*, then*

$$k \leq c(\Gamma) \leq r.$$

Proof.   The lower bound follows readily from Lemma 2.3 and the fact that $k \leq r$ by Fisher's inequality. For the upper bound, we play with r cops. For the initial placement of the cops, choose a point x and place cops on each of the r blocks containing x. Note that since each point occurs in a block with x, the robber cannot begin on a point, or he will be immediately caught. If the robber begins on a block $B = \{y_1, y_2, \ldots, y_k\}$ (which clearly does not contain x), for each $i = 1, 2, \ldots, k$, let B_i be the block containing x and y_i. Note that $B_i \neq B_j$ if $i \neq j$, as otherwise there would be a pair of points in two distinct blocks. Moreover, each block B_i contains a cop, so the cops can move onto the points y_i, $i = 1, 2, \ldots, k$; this guarantees that the robber will be caught in the next round. $\square$

In the case of a projective plane, we have that $k = r$, and so we obtain the exact cop number, which was proved in [19].

Corollary 4.2.   *If Γ is the incidence graph of a projective plane of order q, then* $c(\Gamma) = q + 1$.

Using Theorem 4.1, we obtain new Meyniel extremal families from the following classes of designs.

Theorem 4.3 ([5, 11]).   *Let i and j be integers with $2 \leq i < j$. Then there exist the following designs.*

1. *A* BIBD$(2^{i-1}(2^i - 1), 2^{i-1}, 1)$ *(called an* oval design*)*.
2. *A* BIBD$(2^{i+j} + 2^i - 2^j, 2^i, 1)$ *(called a* Denniston design*)*.

The incidence graphs of oval designs have order $(3)2^{2i-1} - 2^{i-1} - 1$. Since for all y there is a power of 2 between y and $2y$, we use Lemma 3.3 with $x = 2^{i-1}$, $a = 3/2$, $b = d = -1, m = 1$, and X the set of all powers of 2. We consider Denniston designs with $j = i + \alpha$, where α is a constant. In this case, the incidence graphs have order $\beta + \frac{\beta(\beta-1)}{2^i(2^i-1)}$, where $\beta = 2^{i+j} + 2^i - 2^j$. Now apply Lemma 3.3 with $x = 2^i$, $a = 2^\alpha(1 + 2^\alpha)$, $b = 1 + 2^\alpha - 2^{2\alpha}, d = 1 - 2^\alpha, m = 1$, and X the set of powers of 2.

We now turn to a different class of designs which will give Meyniel extremal families. A *GDD* is a triple $(X, \mathcal{G}, \mathcal{B})$ satisfying the following properties:

1. X is a finite set of elements called *points*.
2. $\mathcal{G}$ is a partition of X into subsets called *groups*.
3. $\mathcal{B}$ is a collection of nonempty subsets of X called *blocks*.
4. Each pair of elements in different groups appear in exactly one block.
5. No pair of elements from the same group appear together in a block.

A GDD may be viewed as a decomposition of a complete multipartite graph into cliques. If the blocks have sizes in K, then we speak of a K-GDD; if $K = \{k\}$, then we say it is a k-GDD. The *type* of a GDD is a listing of the sizes of its groups; we use the exponential notation $g_1^{u_1} g_2^{u_2} \cdots g_n^{u_n}$ to indicate that there are u_i groups of size g_i for each $i = 1, 2, \ldots, n$. A k-GDD of type n^k is also referred to as a *transversal design* TD(k, n), which has kn points and n^2 blocks. It is well-known that the existence of a TD$(n + 1, n)$ is equivalent to the existence of a projective plane of order n (and so we know such designs exist for $n = q$ a prime power).

Let us consider the point-block incidence graphs of GDDs. For the purposes of simplification, we first consider only k-GDDs of type n^m.

Lemma 4.4. *The incidence graph of a k-GDD of type n^m has cop number at least*

$$\min \left\{ k, \frac{(m-1)n}{k-1} \right\}.$$

Proof. Let G be the incidence graph of a k-GDD of type n^m. Note that G can contain no 4-cycle, as otherwise there would be a pair of points in two blocks. Since G is bipartite, this means that it has girth at least 6, and so $c(G) \geq \delta(G)$ by Lemma 2.3. Note that G is $\left(k, \frac{(m-1)n}{k-1}\right)$-regular, as vertices of G corresponding to points of the GDD have degree $(m-1)n/(k-1)$ (the number of blocks containing a point), and vertices of G corresponding to blocks of the GDD have degree k. $\square$

In the case of transversal designs, we can find the exact cop number.

Theorem 4.5. *If G is the incidence graph of a* TD(k, n)*, then $c(G) = \min\{k, n\}$.*

Proof. The fact that $c(G) \geq \min\{k, n\}$ is a direct consequence of Lemma 4.4. For the upper bound, we play the game first with k cops, and then with n cops.

Let us first consider the case of k cops. Let the set of cops be $\{C_1, C_2, \ldots, C_k\}$. Let the groups in the GDD be $G_1, G_2, \ldots, G_k$, and for each $i \in \{1, 2, \ldots, k\}$, place C_i on the vertex corresponding to a point $x_i \in G_i$.

Case 1. The robber begins on a vertex corresponding to a point y.

We can clearly assume that $y \notin \{x_1, \ldots, x_k\}$. Without loss of generality, we assume that $y \in G_1$. In the first round, we move the cop C_k onto the block containing $\{y, x_k\}$, and leave the other cops where they are. This forces the robber to move to a block $\{y, y_2, \ldots, y_k\}$ (where $y_i \in G_i$ for each i) to avoid being caught in the next round.

In the second round, we move cop C_i onto the block containing the pair $\{x_i, y_{i+1}\}$ for each $i \in \{1, 2, \ldots, k - 1\}$. This means that the robber is now trapped. If he moves, then he is forced onto one of the points $y, y_2, \ldots, y_k$, each of which is adjacent in G to a vertex containing a cop. But he cannot stay put, since each cop is on a vertex of G adjacent to the one he is currently on.

Case 2. The robber begins on a vertex corresponding to a block $\{y_1, y_2, \ldots, y_k\}$, where $y_i \in G_i$ for each i.

For each $i \in \{1, 2, \ldots, k - 1\}$, move C_i onto the block containing the pair $\{x_i, y_{i+1}\}$, and move C_k onto the block containing the pair $\{x_k, y_1\}$. This creates a scenario as in the second round of Case 1, where the robber is guaranteed to be caught.

Hence, in both cases $c(G) \leq k$.

We now play with n cops. In this case, we begin the game by placing the cops on the points $x_1, x_2, \ldots, x_n$ of part G_1 (say C_i is on x_i). Note that the robber cannot begin the game on a block, as each block contains one of the points $x_1, x_2, \ldots, x_n$. So suppose that the robber begins on point $y \notin G_1$. For each $i \in \{1, 2, \ldots, n\}$, we move C_i to the block containing the pair $\{x_i, y\}$. But each block containing y also contains one of $x_1, x_2, \ldots, x_n$, so the robber cannot move, and will be caught in the next round. □

We have the following corollary.

Corollary 4.6.

1. *If G is the incidence graph of a* TD$(n + 1, n)$, *then* $|V(G)| = (n + 1)n + n^2 = 2n^2 + n$ *and* $c(G) = n$.
2. *If G is the incidence graph of a* TD(k, n), *where $n = k + \alpha$ for some $0 \leq \alpha \leq n$, then* $|V(G)| = (n - \alpha)n + n^2 = 2n^2 - \alpha n$ *and* $c(G) = k = n - \alpha$.

It is interesting to note that a transversal design TD$(n + 1, n)$ is the dual of an affine plane of order n, while if $n = k + \alpha$, where $0 \leq \alpha < n$, then the dual of an affine plane of order n with $\alpha + 1$ parallel classes removed is a TD(k, n). Since a design and its dual have isomorphic incidence graphs, incidence graphs of transversal designs offer an alternative view of the families constructed by Baird [3]. We have, nevertheless, found the exact value of the cop number of these graphs, which was not done in [3].

Next, we consider a further class of GDDs related to transversal designs. A *truncated transversal design* of type $n^k u$, which we will denote by TTD(k, n, u), is a $\{k, k + 1\}$-GDD of type $n^k u$ in which each point in the group of size u occurs only in blocks of size $k + 1$. Rees [21] remarks that the existence of a TTD(k, n, u) is equivalent to the existence of a TD(k, n), which contains u pairwise disjoint parallel classes, or $k - 2$ mutually orthogonal Latin squares of order n with u disjoint common transversals. Note that this implies that $0 \leq u \leq n$ and $n \geq k - 1$. It is easy to see that a TTD(k, n, u) can be constructed from a TD$(k + 1, n)$, by deleting all but u points in a single group.

Theorem 4.7. *If G is the incidence graph of a* TTD(k, n, u), *then*

$$\min\{k, n\} \leq c(G) \leq \min\{k + 1, n\}.$$

Proof. An argument analogous to that in Theorem 4.5 gives the upper bound. For the lower bound, we note that G is C_4-free, as otherwise there would be a pair of points contained in two blocks. Since G is bipartite, Lemma 2.3 gives that $c(G) \geq \delta(G)$. In G, vertices corresponding to blocks of the TTD have degree k or $k + 1$. It is straightforward to check that in a TTD(k, n, u), each point occurs in exactly n blocks, so that the degree of vertices of G corresponding to points of the TTD is n. Thus,

$$c(G) \geq \delta(G) = \min\{k, n\}. \qquad \square$$

Corollary 4.8. *Let G be the incidence graph of a* TTD(k, n, u). *If* $n \in \{k - 1, k\}$, *then* $c(G) = n$. *Otherwise,* $k \leq c(G) \leq k + 1$.

By their construction from transversal designs, a TTD(k, n, u) exists whenever n is a prime power and $3 \leq k \leq n$. The incidence graph of a TTD(k, n, u) has order $n^2 + nk + u$. Thus, in the case $n = k$ (i.e., the TTD is formed by truncating a group in a TD$(n + 1, n)$), the incidence graph of a TTD(n, n, u) has order $2n^2 + u$ for a fixed $0 \leq u \leq n$, and cop number n. If $n = k + \alpha$ for a fixed constant α such that $1 \leq \alpha < n$ and $0 \leq u \leq n$ is fixed, then the incidence graph of a TTD$(n - \alpha, n, u)$ has order $2n^2 - \alpha n + u$ and cop number in $\{n - \alpha, n - \alpha + 1\}$. In either case, letting $k = n - \alpha$, we obtain new Meyniel extremal families for each choice of constants α and u (apply Lemma 3.3 with $a = 2$, $b = -\alpha$, $d = u$, and X the set of primes).

We next consider the incidence graphs of G-designs. Given a fixed graph G, a G-design of order n and index λ is a decomposition of the complete multigraph λK_n into subgraphs, each of which is isomorphic to G. For more details on such designs, see [1] and Section VI.24 of [9]. In the case that each vertex of λK_n appears in a fixed number r of subgraphs, the design is said to be *balanced* or *equireplicate*. In this case, the integer r is called the replication number, and if G has k vertices and e edges, then $r = \lambda(v - 1)k/(2e)$.

Corollary 4.9. *Let G be a graph with k vertices and e edges, and suppose that Γ is the incidence graph of a balanced λ-fold G-design of order v. Suppose that no two distinct blocks intersect in more than x points. If $M = \max\{\lambda, x\}$, then*

$$c(\Gamma) \geq \min\left\{\frac{k}{2M}, \frac{r}{2M}\right\}.$$

Proof. The bound follows immediately from Lemma 2.1, since Γ is $K_{2,M+1}$-free and $\delta(\Gamma) = \min\{k, r\}$. □

Finally, we consider incidence in t-designs. A t-(v, k, λ) *design* is an incidence structure with v points and blocks of size k, such that each t-subset of the point set is contained in exactly λ blocks. Thus, a 2-design is equivalent to a BIBD. In the incidence graph of a t-design, analysis of values of M for which the graph is $K_{2,M+1}$-free does not appear to give any useful information on the cop number. Instead, we define the following graph, which generalizes the incidence graph of a 2-design in a different way. Given a t-(v, k, λ) design and a positive integer $m \leq k$ define its *m-subset incidence graph G_m* as the bipartite graph with bipartition with one part consisting of all m-subsets of the point set, and the other part equalling the set of blocks of the design, where $\{x_1, \ldots, x_m\}$ is adjacent to the block B if and only if $\{x_1, \ldots, x_m\} \subseteq B$. For example, the 1-subset incidence graph is just the incidence graph.

Theorem 4.10. *If G_{t-1} is the $(t - 1)$-subset incidence graph of a t-$(v, k, 1)$ design, then we have that*

$$c(G_{t-1}) \geq \min\left\{\binom{k}{t - 1}, \frac{v - t + 1}{k - t + 1}\right\}.$$

Proof. Note that G_{t-1} is $\left(\binom{k}{t-1}, \frac{v-t+1}{k-t+1}\right)$-regular, so by Lemma 2.3, and since G_{t-1} is bipartite, it suffices to show that G_{t-1} contains no 4-cycle. A 4-cycle in G_{t-1} would

correspond to two $(t-1)$-subsets S_1 and S_2 of the point set of the design, which are contained in a common pair of blocks, B_1 and B_2. But given two distinct $(t-1)$-subsets S_1 and S_2 of the point set, the number of points in $S_1 \cup S_2$ is at least t. Hence, the number of blocks containing $S_1 \cup S_2$ can be at most 1. $\qquad\square$

Of course, if $t = 2$, then G_{t-1} is the ordinary incidence graph and the bound is the same as the lower bound given in Theorem 4.1. For $t \geq 3$, we are unaware of any new Meyniel extremal families from Theorem 4.10, though we can find families of graphs with unbounded cop number when we consider *spherical geometries*. For a prime power q, these are 3-$(q^d + 1, q + 1, 1)$ designs. When $d = 2$, they are the *inversive planes*, *Möbius planes*, or *circle geometries* (see Example 4.30 in [9], and also [10]). In this case, the order of the incidence graph is $\binom{q^2+1}{2} + \binom{q^2+1}{3}/\binom{q+1}{3} \sim q^4$ and the cop number is at least $q + 1$.

5. BLOCK INTERSECTION GRAPHS

Given a design $X = (V, \mathcal{B})$, its *block intersection graph*, written $\mathrm{BIG}(X)$, has vertex set $\mathcal{B}$, with B_1 and B_2 adjacent if and only if $B_1 \cap B_2 \neq \emptyset$. In general, it seems determining the exact cop number of a block intersection graph of a $\mathrm{BIBD}(v, k, \lambda)$ may not be trivial. Since block intersection graphs of BIBDs (and t-designs in general) have diameter (at most) 2, they satisfy Meyniel's conjecture; by the upper bound for such graphs of Lu and Peng [15], if G is the BIG of a $\mathrm{BIBD}(v, k, \lambda)$, then

$$c(G) \leq 2\sqrt{|V(G)|} - 1 = 2\sqrt{\frac{\lambda v(v-1)}{k(k-1)}} - 1.$$

The following result improves this bound when v is relatively large with respect to k.

Lemma 5.1.   *If X is a t-(v, k, λ) design, then $c(\mathrm{BIG}(X)) \leq \lceil k/(t-1) \rceil$. In particular, if X is a $\mathrm{BIBD}(v, k, \lambda)$, then $c(\mathrm{BIG}(X)) \leq k$.*

Proof.   Choose a point y, and place each cop on a block containing y. The robber begins on a block $B = \{x_1, x_2, \ldots, x_k\}$. We may assume that this block is disjoint from each block containing a cop, as otherwise, the robber will be immediately caught. Partition the elements of B into $\lceil k/(t-1) \rceil$ subsets $B_1, B_2, \ldots, B_{\lceil k/(t-1) \rceil}$, where $|B_i| \leq t - 1$ for each i. In the first round, we may move a cop onto a block containing $\{y\} \cup B_i$ for each $1 \leq i \leq \lceil k/(t-1) \rceil$. But then any move by the robber places him on a block that intersects one of the blocks containing a cop. $\qquad\square$

The upper bound from Lemma 5.1 is not attained for certain designs. In fact, if we are looking for a relatively large cop number, the projective and affine planes are not good examples in this context.

Lemma 5.2.

1. *The block intersection graph of a projective plane is a complete graph, and so has cop number 1.*
2. *The block intersection graph of an affine plane on q^2 points is a complete multipartite graph with $q + 1$ parts of size q, and consequently its cop number is 2.*

For block intersection graphs of BIBDs, we cannot use a girth or C_4-free argument to determine any lower bound on the cop number.

Theorem 5.3. *[16] The block intersection graph G of a* BIBD(v, k, λ) *is pancyclic; that is, it contains a cycle of each length $3, 4, \ldots, |V(G)|$.*

If the order is sufficiently large with respect to the block size and $\lambda = 1$, then we have the following lower bound.

Lemma 5.4. *Let G be the block intersection graph of a* BIBD$(v, k, 1)$*, where $v > k(k - 1)^2 + 1$. Then $c(G) \geq k$.*

Proof. We play with $(k - 1)$-many cops. Consider the first round of the game, where the cops have been placed on blocks $C_i = \{c_{i1}, c_{i2}, \ldots, c_{ik}\}, i = 1, 2, \ldots, k - 1$. Choose a point $x_1 \notin \bigcup_{i=1}^{k-1} C_i$, which is possible since $v > k(k - 1)$. Note that the number of blocks containing x_1 and some element of $\bigcup_{i=1}^{k-1} C_i$ is at most $k(k - 1)$, while the total number of blocks containing x_1 is $r = (v - 1)/(k - 1)$. Hence, if $r > k(k - 1)$, that is, $v > k(k - 1)^2 + 1$, then the robber can move onto the block $X = \{x_1, x_2, \ldots, x_k\}$, and be safe heading into the next round.

Now, suppose that we have completed the ith round, and the robber is currently safe on the block $Y = \{y_1, y_2, \ldots, y_k\}$; that is, Y is disjoint from each block containing a cop. After the cops move in round $i + 1$, suppose they are on blocks $C'_i = \{c'_{i1}, c'_{i2}, \ldots, c'_{ik}\}$, $i = 1, 2, \ldots, k - 1$. Note that each of $C'_1, C'_2, \ldots, C'_{k-1}$ can contain at most one element of Y, as otherwise there is a pair of elements in Y, which also occurs in one of the blocks $C'_1, C'_2, \ldots, C'_{k-1}$. Hence, $\bigcup_{i=1}^{k-1} C'_i$ contains at most $(k - 1)$ of the elements of Y. Without loss of generality, suppose that $y_1 \notin \bigcup_{i=1}^{k-1} C'_i$. A similar argument to before shows that there is a block containing y_1 which is disjoint from $\bigcup_{i=1}^{k-1} C'_i$; the robber may move onto this block and not be caught in the round $i + 1$. The result follows by induction. □

Note that a projective plane of order q is a BIBD$(q^2 + q + 1, q + 1, 1)$; in this case $q^2 + q + 1 < q^2(q + 1) + 1$, so the lower bound does not apply. Similarly, affine planes do not have enough points for Lemma 5.4 to apply.

In light of Lemmas 5.1 and 5.4, we can state the cop number when the order is sufficiently large with respect to the block size in the case of designs of index 1.

Corollary 5.5. *If G is the block intersection graph of a* BIBD$(v, k, 1)$*, where $v > k(k - 1)^2 + 1$, then $c(G) = k$.*

The argument in the proof of Lemma 5.4 does not work when $\lambda > 1$. We can still place the robber in the initial round so that he cannot be caught (the number of points needed would also be $v > k(k - 1)^2 + 1$), but we cannot guarantee his safety in subsequent rounds.

6. POINT GRAPHS

Given an incidence structure $X = (V, \mathcal{B})$, its *point graph* is a graph G with vertex set V, such that two points are adjacent if they occur in a common block in $\mathcal{B}$. Note that the point graph of a BIBD(v, k, λ) is a complete multigraph, and so has cop number 1. We thus consider point graphs of packings, or partial designs. A t-(v, k, λ) *packing design* is an incidence structure with v points and blocks of size k, such that each t-subset of the point set is contained in at most λ blocks; we will consider only the case that $t = 2$ and $\lambda = 1$. A *resolution class* in a packing design is a set of blocks, which partition the point set. If the blocks of a packing can be partitioned into resolution classes, then the packing is said to be *resolvable*.

Theorem 6.1.   *Let $t \geq 0$ be a fixed integer. Suppose X is a resolvable 2-$(v, k, 1)$ packing design with $t + 1$ resolution classes, and let P be the point graph of X. If, for some fixed integer j:*

  1. $j((q - 1)(t + 1) + 1) < v$
  2. $j < t + 1$
  3. $tj < k$,

then $c(P) \geq j + 1$.

Proof.   Suppose we play with j cops. For any vertex x of P, note that the degree of x is $(t + 1)(k - 1)$. Thus, once the cops have been placed, the closed neighbor set of the vertices containing the cops has size at most $j + (k - 1)(t + 1)j$. Since $v > j + (k - 1)(t + 1)j$ by item (1), the robber is guaranteed to be able to choose a starting vertex in the first round, which is not adjacent to a cop.

Now, suppose that we are in the ith round, the cops have just moved, and the robber is on vertex R. Since the number of blocks containing R is $t + 1$ and since $t + 1 > j$, by (2), there is guaranteed to be a block B containing R that contains no cops. Note that if a cop is on vertex C, there are at most t points on B, which occur in a block with C (one for each parallel class other than the one containing B); thus, there are at most tj points on B adjacent to a cop. Since the number of points of B is greater than tj by (3), the robber can choose a point on B to move to which is not adjacent to any cop, so that he is not caught on the cops' next move in round $i + 1$. Hence, if we play with j cops, the robber avoids capture for all i by an induction on i.   □

In the particular case that the design is an affine plane, we obtain the following.

Corollary 6.2.   *Let $t \geq 0$ be a fixed integer. Suppose A is an affine plane of order q, where $q + 1 > t + 1$, and delete the lines in all but $t + 1$ parallel classes to form a design X. Let P be the point graph of X. If, for some fixed integer j:*

  1. $j((q - 1)(t + 1) + 1) < q^2$
  2. $j < t + 1$
  3. $tj < q$,

then $c(P) \geq j + 1$.

For example, if q is a sufficiently large prime power, $t + 1 \sim \sqrt{q}$ and $j \sim \sqrt{q} - 1$, then conditions (1), (2), and (3) hold, giving a lower bound on $c(P)$, which is asymptotic

to $n^{1/4}$, where $n = q^2$ is the order of the graph P. Note that while this does not give a Meyniel extremal family, it does give one with unbounded cop number.

The type of partial affine plane formed by deleting parallel classes is an example of a *partial geometry*, which we now define. A *partial geometry* pg(s, t, α) is an incidence structure with lines of size $s + 1$, such that each point is on $t + 1$ lines and for each point p and line L with p not on L, there are α lines through p, which intersect L. The *dual* of a partial geometry is the point-line incidence structure obtained by switching the points and lines. Partial geometries can be divided into four classes:

1. A partial geometry with $s + 1 = \alpha$ is a 2-$(v, s + 1, 1)$ design. The dual satisfies $t + 1 = \alpha$ and is a called a *dual design*.
2. A partial geometry with $t = \alpha$ is called a *net*. The dual satisfies $s = \alpha$ and is a transversal design.
3. A partial geometry with $\alpha = 1$ is called a *generalized quadrangle*.
4. If $1 < \alpha < \min\{s, t\}$, then the partial geometry is called *proper*.

For more on partial geometries and their point graphs, see [6, 22]. The following result may be proved in an analogous manner to Lemma 6.1.

Theorem 6.3.   *Suppose G is the point graph of a* pg(s, t, α). *If the following conditions hold:*

1. $j(s(t + 1) + 1) < (s + 1)(st + \alpha)/\alpha$
2. $j < t + 1$
3. $j\alpha < s + 1$,

then $c(G) \geq j + 1$.

For instance, in the case of a generalized quadrangle satisfying conditions (1)–(3) of Theorem 6.3, applying the theorem with $j = \min\{s, t\}$ gives that the cop number of the point graph is at least $\min\{s + 1, t + 1\}$. For example, consider the generalized quadrangle with $s = t = q$. This is the *parabolic quadric* $Q(q, 4)$, which consists of points of P$(4, q)$ that are singular with respect to a nondegenerate quadratic form on P$(4, q)$, which is, up to a coordinate transformation, unique. The conditions of Theorem 6.3 are met in this case with $j = q$, giving a cop number of at least $q + 1$, which is approximately $n^{1/3}$, where n is the order of the point graph.

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