ALL COUNTABLE MONOIDS EMBED INTO THE MONOID OF THE INFINITE RANDOM GRAPH

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Abstract. We prove that the endomorphism monoid of the infinite random graph R contains as a submonoid an isomorphic copy of each countable monoid. As a corollary, the monoid of R does not satisfy any non-trivial semigroup identity. We also prove that the full transformation monoid on a countably infinite set is isomorphic to a submonoid of the monoid of R.

1. Introduction

The infinite random graph, written R, is the unique (up to isomorphism) countable graph that satisfies the *existentially closed* adjacency property: for all finite disjoint sets of vertices A and B, there is a vertex joined to each vertex of A and not joined nor equal to a vertex of B. The graph R has many remarkable properties which have attracted the attention of several researchers, including graph theorists, logicians, and algebraists. One such property, known to Fraïssé [7] in 1953, is that R is a *universal* graph: each countable graph is isomorphic to an induced subgraph of R. We note that the term "universal graph" has several meanings in graph theory. Sometimes it is restricted to finite graphs (see [5]) and sometimes the class of subgraphs need not be induced (see [8]). For other properties of R, the reader is directed to the surveys of P. Cameron [3, 4].

While the automorphism group of R has been thoroughly investigated (see the references in [3]), properties of the endomorphism monoid, or monoid, of R have been largely overlooked. In [2], the first two authors initiated the study of the monoid of R, written $\mathrm{End}(R)$, and characterized properties of its retracts. The monoid $End(R)$ was further studied in [1, 6].

1991 *Mathematics Subject Classification.* 05C80, 20M20.

Key words and phrases. graph, monoid, graph homomorphism, embedding, infinite random graph.

The first two authors gratefully acknowledge support from the Natural Science and Engineering Research Council of Canada (NSERC).

We prove in the present article that End(R) is universal as a monoid: that is, it contains as a submonoid an isomorphic copy of each countable monoid; see Theorem 3. We say End(R) *embeds* each countable monoid. We investigate some of the computational consequences of this result in Corollaries 4 and 5, where we prove that End(R) satisfies no non-trivial semigroup identity, and has undecidable universal theory, respectively. In Theorem 6 we prove that the full transformation monoid on a countable set embeds in End(R). This result gives a second proof that End(R) is universal. For further background and notation the reader is directed to [2]. Unless otherwise stated, all graphs and semigroups considered are countable, and we consider only simple undirected graphs.

2. The main result and its consequences

The following representation theorem (which, to our knowledge, is a part of folklore) is fundamental to our approach.

Theorem 1. *If S is a monoid of cardinality κ, then there is a graph G such that S embeds in* End(G) *and* $|V(G)| \leq \aleph_0 + \kappa$.

A graph G is *algebraically closed* if for each finite subset S of $V(G)$, there is a vertex z not in S that is joined to each vertex of S. The following result is Proposition 4.2 of [2], which we restate here for completeness.

Theorem 2. *Let H be a graph. There is $f \in E(\mathrm{End}(R))$ with $R \restriction Im(f) \cong H$ if and only if H is a countable algebraically closed graph.*

We will use Theorem 2 in the proof of the following result, which is our main theorem.

Theorem 3. *If S is a countable semigroup, then S embeds in* End(R).

Proof. Since each semigroup S may be embedded in a monoid with cardinality at most $|S| + 1$, we may assume without loss of generality that S is a monoid. By Theorem 1, let G be a countable graph such that S embeds in End(G). Let

$$G^+ = G \vee K_{\aleph_0},$$

the graph formed by adding all edges between copies of G and $K_{\aleph_0}$. As G^+ is algebraically closed, and since R is a universal graph, by Theorem 2 there is a retract $\alpha \in E(\mathrm{End}(R))$ with $Im(\alpha) = V(G^+)$.

Define $\varphi : \mathrm{End}(G^+) \to \mathrm{End}(R)$ by

$$\varphi(f) = \begin{cases} f\alpha & \text{if } f \neq 1_{G^+} \\ 1_R & \text{else.} \end{cases}$$

If $f, g \in \mathrm{End}(G^+)$ are distinct, then $\varphi(f)$ and $\varphi(g)$ are distinct. To see this, first consider the case when neither f nor g are equal to 1_{G^+}. Note that if $x \in V(G^+)$, then $\varphi(f)(x) = f\alpha(x) = f(x)$ as α is a retract; similarly, $\varphi(g)(x) = g(x)$. Hence, $\varphi(f) \neq \varphi(g)$ in this case. Suppose now that exactly one of f or g equals 1_{G^+}; say $f = 1_{G^+}$ and $g \neq f$. By the existentially closed property, R is not isomorphic to the join of two countable graphs. In particular, $R \ncong G^+$. Hence, we may choose some $x \in V(R)\backslash V(G^+)$. Then $\varphi(f)(x) = x$, while $\varphi(g)(x) = g\alpha(x) \in V(G^+)$. Hence, $\varphi(f) \neq \varphi(g)$ in this case as well.

Now, fix $f, g \in \mathrm{End}(G^+)$. First consider the case when neither f nor g equal 1_{G^+}. Then $\varphi(fg) = fg\alpha$, and $\varphi(f)\varphi(g) = f\alpha g\alpha$. Let x be a fixed vertex of R. Consider $f\alpha g\alpha(x)$. Then $\alpha(x) \in V(G^+)$, and so $g\alpha(x) \in V(G^+)$. But since α is the identity on G^+, we have that $\alpha g\alpha(x) = g\alpha(x)$. Hence,

$$\varphi(f)\varphi(g)(x) = f\alpha g\alpha(x) = fg\alpha(x) = \varphi(fg)(x).$$

Suppose now that exactly one of f or g equals 1_{G^+}. Consider the case when $f = 1_{G^+}$ and $g \neq f$. (The other case is similar, and so is omitted.) Then $\varphi(fg) = \varphi(1_{G^+}g) = \varphi(g)$, while $\varphi(f)\varphi(g) = 1_R\varphi(g) = \varphi(g)$. Hence, φ is an embedding of monoids.

For a fixed $f \in \mathrm{End}(G)$, define $\psi(f) = f \cup 1_{K_{\aleph_0}}$. That is, $\psi(f)$ is the self-mapping on G^+ that is f on G and the identity on $K_{\aleph_0}$. By the properties of join for graphs, it is not hard to see that $\psi(f) \in \mathrm{End}(G^+)$. Hence, $\psi : \mathrm{End}(G) \to \mathrm{End}(G^+)$ is well-defined, and the reader may verify that it is injective.

For $f, g \in \mathrm{End}(G)$, $\psi(fg) = fg \cup 1_{K_{\aleph_0}}$, and

$$\psi(f)\psi(g) = (f \cup 1_{K_{\aleph_0}})(g \cup 1_{K_{\aleph_0}}).$$

To prove that ψ is an endomorphism, we consider cases for the location of $x \in V(G^+)$.

If $x \in V(G)$, then

$$\begin{aligned} \psi(fg)(x) &= (fg \cup 1_{K_{\aleph_0}})(x) = fg(x) \\ &= (f \cup 1_{K_{\aleph_0}})(g \cup 1_{K_{\aleph_0}})(x) \\ &= \psi(f)\psi(g)(x), \end{aligned}$$

since $(g \cup 1_{K_{\aleph_0}})(x) = g(x) \in V(G)$. If $x \in V(K_{\aleph_0})$, then

$$\begin{aligned} \psi(fg)(x) &= (fg \cup 1_{K_{\aleph_0}})(x) \\ &= x = (f \cup 1_{K_{\aleph_0}})(g \cup 1_{K_{\aleph_0}})(x) \\ &= \psi(f)\psi(g)(x). \end{aligned}$$

As x is arbitrary, we have that $\psi(fg) = \psi(f)\psi(g)$, and so ψ is an embedding of semigroups. The map ψ is an embedding of monoids since $\psi(1_G) = 1_{G^+}$. To finish the proof, observe that $\psi\varphi : \mathrm{End}(G) \to \mathrm{End}(R)$ is an embedding of monoids. □

Theorem 3 has the following computational consequences. We refer the reader to Hodges [9] for any terms not explicitly defined.

Corollary 4. *The monoid* End(R) *does not satisfy any non-trivial semigroup (monoid) identity. In particular,* End(R) *generates the variety of all semigroups (monoids).*

Proof. Since every countable semigroup embeds into End(R) by Theorem 3, so does the free semigroup on a countable set of generators, written $F_{Sem}(X)$. If there were an equation $s = t$ in the language of semigroups that is not consequence of the associative law, and satisfied by End(R), then $s = t$ would be satisfied by $F_{Sem}(X)$, which is a contradiction. □

Corollary 5. *The universal theory of* End(R) *is undecidable.*

Proof. We first observe that the universal theory of End(R) equals the universal theory of all semigroups. To see this, note that since every countable semigroup embeds into End(R) by Theorem 3, every universal sentence true in End(R) will be true in all countable semigroups and, by the Löwenheim-Skolem Theorem (see [9]), in all semigroups.

It is well-known that the universal theory of semigroups is undecidable. (See [11, 12, 13]. This is implied by the existence of a semigroup with undecidable word problem.) Hence, the universal theory of End(R) is undecidable. □

3. The full transformation monoid

If X is a nonempty set, then a *self-mapping* of X is a mapping from X to X. The *full transformation monoid*, written $T(X)$, is the monoid of all self-mappings of X under composition. We prove in Theorem 6 below that $T(X)$, where X is countably infinite, embeds in End(R). As it is well-known that every countable semigroup embeds in $T(X)$, this gives a second proof of Theorem 3. The proof we give of Theorem 6 has the advantage over our proof of Theorem 3 that it does not rely on results from logic nor on results from [2]. The monoid End(R) (which has cardinality $2^{\aleph_0}$) does not embed all monoids of cardinality at most $2^{\aleph_0}$. The reason for this is that $T(X)$, where X is countable, does not embed all monoids of cardinality at most $2^{\aleph_0}$, and by Theorem 6, $T(X)$ and End(R) are mutually embeddable. For example, an uncountable direct sum of countable simple groups does not embed into $T(X)$ [10]. We do not know, however, exactly which uncountable monoids embed in End(R).

If $(G_n : n \in \mathbb{N})$ is a sequence of graphs with $G_n \leq G_{n+1}$, then define

$$\lim_{n\to\infty} G_n = \bigcup_{n\in\mathbb{N}} G_n;$$

we call $\lim_{n\to\infty} G_n$ the *limit* of the sequence $(G_n : n \in \mathbb{N})$. We say that $(G_n : n \in \mathbb{N})$ is a *chain* of graphs.

Theorem 6. *The transformation monoid* $T(X)$, *where* X *is countable, embeds in* End(R).

Proof. We define a countably infinite graph R^* by induction on $n \in \mathbb{N}$. Let R_0 be an isomorphic copy of $\overline{K_{\aleph_0}}$. Assume that R_n has been defined so that R_0 is an induced subgraph of R_n, and $V(R_n)$ is countable. To define $R_{n+1,}$ for each finite subset $S \subseteq V(R_n)$, add a new vertex $x_S \notin V(R_n)$ so that x_S is joined to exactly the vertices of S.

As $(R_n : n \in \mathbb{N})$ is a chain, we define $R^* = \lim_{n\to\infty} R_n$. It is not hard to see that R^* is existentially closed; hence, $R^* \cong R$. We identify R^* with R in what follows.

Since each self-mapping of $\overline{K_{\aleph_0}}$ is an endomorphism, we have that $\mathrm{End}(R_0) \cong T(X)$. Fix $f \in \mathrm{End}(R_0)$; let $f = f_0$. Assume that $f_n \in \mathrm{End}(R_n)$ is defined, and $f_n \restriction R_0 = f_0$. Define

$$f_{n+1}(z) = \begin{cases} x_{f_n(S)} & \text{if } z = x_S \\ f_n(z) & \text{if } z \in V(R_n). \end{cases}$$

The mapping f_{n+1} *extends* f_n; that is, $f_{n+1} \restriction R_n = f_n$. To see that f_{n+1} is a homomorphism, by inductive hypothesis, and since any two vertices $x, y \in V(R_{n+1}) \setminus V(R_n)$ are not joined, we need only consider the case when $x \in V(R_{n+1}) \backslash V(R_n)$, $y \in V(R_n)$, and xy is an edge of R_{n+1}. By construction, x is the vertex $x_S \in V(R_{n+1}) \backslash V(R_n)$ for some unique finite subset $S \subseteq V(R_n)$; hence, $y \in S$. But then

$$f_{n+1}(x) f_{n+1}(y) = x_{f_n(S)} f_n(y) \in E(R_{n+1}),$$

since $f_n(y) \in f_n(S)$ and $x_{f_n(S)}$ is joined to each vertex of $f_n(S)$.

Define $F = \bigcup_{n\in\mathbb{N}} f_n$. Then F extends f and is an endomorphism of R. Define $\varphi : T(X) \to \mathrm{End}(R)$ by $\varphi(f) = F$. Since F extends f, we have that φ is injective. To complete the proof, we now show that φ is an embedding of monoids.

Fix $f, g \in T(X) = \mathrm{End}(R_0)$. It is sufficient to prove that $\varphi(fg)_n = \varphi(f)_n \varphi(g)_n$ for all $n \in \mathbb{N}$. This holds for $n = 0$, so we proceed by induction on n. If we consider the case for $n + 1$, then we must show that $\varphi(fg)_{n+1}(x) = \varphi(f)_{n+1} \varphi(g)_{n+1}(x)$, for each $x \in V(R_{n+1})$. By inductive hypothesis, we may assume that $x \notin V(R_n)$. Hence, x is some vertex $x_S \in V(R_{n+1}) \setminus V(R_n)$, where $S \subseteq V(R_n)$ is finite and unique. Then

$$\begin{aligned} \varphi(fg)_{n+1}(x) &= (fg)_{n+1}(x) = x_{(fg)_n(S)} = x_{f_n(g_n(S))} = f_{n+1}(x_{g_n(S)}) \\ &= f_{n+1}g_{n+1}(x) = \varphi(f)_{n+1}\varphi(g)_{n+1}(x), \end{aligned}$$

where the third equality follows by the inductive hypothesis. □

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