THE STRUCTURE AND AUTOMORPHISMS OF SEMI-DIRECTED GRAPHS

ANTHONY BONATO, DEJAN DELIĆ, AND CHANGPING WANG

Abstract. Complex real-world networks such as the web graph are often modelled as directed graphs evolving over time, where new vertices are joined to a constant m number of existing vertices of prescribed type. We consider a certain on-line random construction of a countably infinite graph with out-degree m, and show that with probability 1 the construction gives rise to a unique isomorphism type. We show that random semi-directed graphs are prime models in a certain first-order theory. We study algebraic properties of random semi-directed graphs; in particular, we prove that their automorphism groups embed all countable groups.

1. Introduction

Complex networks arise in many real-world contexts, ranging from the web graph to networks arising in the biological and social sciences. Such networks are usually modelled as directed graphs (digraphs) that evolve over time, where new vertices and edges are born over time. A large number of stochastic models for complex networks have been proposed; see [4, 13] for a discussion of such models. For example, in the *preferential attachment model* introduced in [1, 2], new vertices are born over time which have a greater probability to join to high degree vertices. The digraphs generated by the preferential attachment models have the property that each vertex has exactly m out-neighbours. Constant out-degree is in fact, a common assumption in other models of complex networks; see, for example, [1, 2, 13, 20]. Hence, models of complex networks often generate directed graphs satisfying the following properties.

(1) *On-line*: digraphs are generated over a countably infinite set of discrete time-steps, with a countable (either finite or countably infinite) set of vertices born at each time-step. At time 0, a fixed *initial digraph* H is given.
(2) *Constant out-degree*: new vertices have edges directed only to existing vertices, and for $m > 0$ a fixed integer, there are exactly m such edges.

A digraph G satisfying these two properties is called *semi-directed with initial graph H and constant out-degree m*; we sometimes refer to G simply as *semi-directed.* The moniker "semi-directed" comes from [3] (see p. 17). It emphasizes

1991 *Mathematics Subject Classification.* 05C75, 20B27, 03C15, 05C80.

Key words and phrases. universal graphs, semi-directed graphs, web graph, adjacency property, automorphism group, prime model, undecidable universal theory.

The authors gratefully acknowledge support from NSERC and MITACS.

that the orientation of edges in a semi-directed graph canonically arises according to time: new vertices may only point to vertices born at earlier time-steps. Note that semi-directed graphs have no infinite directed path emanating from any vertex, are connected, and is acyclic so long as H is acyclic.

We consider the infinite semi-directed limit graphs that result when time tends to infinity. Analyzing stochastic models by considering the infinite limit is a common technique in the natural sciences. In particular, the existence of a unique limit indicates coherent behaviour of the model, while many distinct limits suggest a sensitivity to initial conditions that is an indicator of chaos. In [7, 8, 19], infinite limits of graphs generated by models of the web graph were investigated. Limits generated by on-line graph processes were, in fact, studied by Fraïssé [16] and others decades prior to birth of the internet.

One of the most studied examples of an infinite limit graph arising from a stochastic model is the infinite random graph. The probability space $G(\mathbb{N}, p)$ consists of graphs with vertices $\mathbb{N}$, so that each distinct pair of integers is joined independently with a fixed probability $p \in (0, 1)$. Erdős and Rényi discovered that with probability 1, all $G \in G(\mathbb{N}, p)$ are isomorphic. The unique isomorphism type of countably infinite graph is named the *infinite random graph*, or the *Rado* graph, and is written R; see the survey [10].

Define a deterministic graph R^* as follows. Let R_0 be the graph with one vertex, K_1. Assume that for a nonnegative integer $t \geq 0$, the graph R_t is defined and finite. To form R_{t+1}, for each subset $S \subseteq V(R_t)$ (possibly empty) add a vertex z_S joined only to the vertices of S. The sets $\{V(R_t) : t \in \mathbb{N}\}$ and $\{E(R_t) : t \in \mathbb{N}\}$ are well-ordered sets or *chains*. We define

$$V(R^*) = \bigcup_{t \in \mathbb{N}} V(R_t), \quad E(R^*) = \bigcup_{t \in \mathbb{N}} E(R_t).$$

We write $\lim_{t \to \infty} R_t = R^*$, and say that R^* is the *limit of the chain* $(R_t : t \in \mathbb{N})$. The notion of limit extends to any chain $(G_t : t \in \mathbb{N})$ of graphs.

A graph G is *existentially closed* or *e.c.* if for all finite disjoint sets of vertices A and B (one of which may be empty), there is a vertex $z \notin A \cup B$ joined to all of A and to no vertex of B. By a back-and-forth argument, $R \cong R^*$ is the unique isomorphism type of countably infinite graphs that is e.c. Further, R is a *universal* graph: it contains as an induced subgraph an isomorphic copy of each countable graph.

In the present article, we consider structural, logical, and algebraic properties of certain infinite semi-directed graphs that arise naturally as limits of on-line random processes. The current article is the full version of [6]. Analogous to R, these so-called *random semi-directed* graphs have isomorphism types characterized via a set of adjacency properties (see Theorem 1). As an application of the characterization, random semi-directed graphs are shown to be universal (see Corollary 2). Random semi-directed graphs are generated with probability 1 by a natural stochastic process (see Corollary 4). We explore some of the first-order properties of semi-directed graphs. While semi-directedness is not a first-order property, we show in Theorem 5 that the random semi-directed graphs form a class of prime models in a suitable first-order theory. The automorphism group of R has been thoroughly investigated (see [10]). In Section 3 we show that all countable groups embed in the group of a random semi-directed graph. This property of the automorphism group of random graph parallels the universality result of Corollary 2. The universality of the automorphism group of random semi-directed graphs is used to show in Corollary 9 that their universal theories are undecidable.

All graphs we consider are simple (that is, no loops nor multiple edges), directed, and countable. If (x, y) is a directed edge, then y is an *out-neighbour* of x. We say that G *embeds in* H and write $G \leq H$ if G is isomorphic to an induced subdigraph of H. If $S \subseteq V(G)$, then we write $G[S]$ for the subdigraph induced by S (we omit the subscript G if it is clear from context). An *independent set* of vertices is a set of vertices with no directed edges between them. The vertices of a semi-directed graph with initial graph H may be ordered in the following way. A vertex x not in H has *height* k if there is a directed path of length k from x ending in a vertex of H, and there is no such path of length less than k. Vertices in H have height 0. The height of a finite set S of vertices is the maximum height of a vertex in S. For background in graph theory, the reader is directed to [14, 24].

The *automorphism group* (or *group*) of G is written $\mathrm{Aut}(G)$. We write $\mathbb{N}$ for the natural numbers, $\mathbb{N}^+$ for the positive integers, and $\aleph_0$ for the cardinality of $\mathbb{N}$.

2. Random semi-directed graphs

We consider the following general framework for limits of semi-directed graphs. A class $\mathcal{C}$ of digraphs closed under isomorphism is *good* if it both contains infinitely many digraphs, and is *hereditary*: if $G \in \mathcal{C}$ and $H \leq G$, then $H \in \mathcal{C}$. For example, the class of all digraphs is good, as the class of linear orders (that is, transitive tournaments).

For the remainder of the article, fix $m > 0$ an integer, $\mathcal{C}$ a good class of digraphs, and H an m-vertex digraph in $\mathcal{C}$ (which exists as $\mathcal{C}$ is good). We define a countably infinite graph $R_{m,H}(\mathcal{C})$ as follows. Let R_0 be H. Assume that R_t is defined and countable so that $R_0 \leq R_t$. To form R_{t+1}, for each induced subdigraph S of R_t that has m vertices and is in $\mathcal{C}$, add a vertex x_S that is joined to each vertex of S (that is, there are directed edges from x_S to each vertex of S) and no other vertices in R_t. Define $R_{m,H}(\mathcal{C}) = \lim_{t\to\infty} R_t$. The countably infinite digraph $R_{m,H}(\mathcal{C})$ is semi-directed by its construction. The idea behind the definition of $R_{m,H}(\mathcal{C})$ is that all m-sets of vertices S that induce a graph in $\mathcal{C}$ are extended: the vertex x_S has its out-neighbours equalling S. We say that the vertex x_S was *born at time* t. Observe that vertices born at time t have height t.

One of our main results is that the isotype of $R_{m,H}(\mathcal{C})$ may be captured by a set of simple set of properties. We say that a digraph G is $(\mathcal{C}, m)$-*e.c.* if for each set A of m-vertices which induces a graph in $\mathcal{C}$ and each finite set B of vertices disjoint from A, there is a vertex $z \notin A \cup B$ so that $(z, a) \in E(G)$ for all a in A, but there are no directed edges between z and vertices of B. The $(\mathcal{C}, m)$-e.c. property is a directed analogue of the e.c. property, relativized by the parameter m and by the restriction that $G[A] \in \mathcal{C}$. We note that (assuming that the class $\mathcal{C}$ is axiomatized by finitely many first-order sentences), that the $(\mathcal{C}, m)$-e.c. property is first-order expressible in the language of graphs; see Subsection 2.2.

Theorem 1. *A countable digraph G is isomorphic to $R_{m,H}(\mathcal{C})$ if and only if G is semi-directed with initial graph H and constant out-degree m, each out-neighbour set induces a subdigraph in $\mathcal{C}$, and G is $(\mathcal{C}, m)$-e.c.*

Proof. As the forward direction is immediate, we prove only the reverse direction. Let H' be the initial copy of H in G. The set $V(G)\backslash V(H')$ has a *topological sort*: an enumeration $(x_t : t \in \mathbb{N}^+)$ of $V(G)\backslash V(H')$ with the property that if (x_i, x_j) is a directed edge, then $i > j$. To see this, we may choose x_1 to be any vertex with height 1. Assuming that $\{x_1, \ldots, x_t\}$ were chosen, consider a vertex u of $V(G)\backslash(V(H') \cup \{x_1, \ldots, x_t\})$. If u has out-degree 0 in $G[V(G)\backslash(V(H') \cup \{x_1, \ldots, x_t\})]$, then let $x_{t+1} = u$. Otherwise, in $G[V(G)\backslash(V(H') \cup \{x_1, \ldots, x_t\})]$ there is a maximal directed finite path from u. The end point v of this path has out-degree 0, and we choose $x_{t+1} = v$.

As G is arbitrary with the given properties, it follows that $R_{m,H}(\mathcal{C})$ also has a topological sort. Now, let $(x_t : t \in \mathbb{N}^+)$ and $(y_t : t \in \mathbb{N}^+)$ be topological sorts of $V(G)\backslash V(H')$ and $V(R_{m,H}(\mathcal{C}))\backslash V(R_0)$, respectively. We proceed by a back-and-forth argument, with f_0 isomorphically mapping H' in G to H at time 0 in $R_{m,H}(\mathcal{C})$. For $t \geq 0$, suppose that f_t is a partial isomorphism with domain X_t containing $V(H') \cup \{x_1, \ldots, x_t\}$ and range Y_t containing $V(R_0) \cup \{y_1, \ldots, y_t\}$. We will assume as an additional inductive hypothesis that X_t and Y_t are *closed*: all out-neighbours of vertices in the set are in the set itself.

Suppose first that $t + 1 \geq 1$ is odd. In this case, we go forward. Let x be the lowest indexed vertex of $(x_t : t \in \mathbb{N})$ not in X_t. As the enumeration is a topological sort, the set of out-neighbours S_t of x are in X_t; as X_t is closed, no vertex of X_t points to x. By hypothesis, $|S_t| = m$ and $G[S_t] \in \mathcal{C}$. As $R_{m,H}(\mathcal{C})$ satisfies the $(\mathcal{C}, m)$-e.c. property, there is a vertex y whose out-neighbours are exactly $f_t(S_t)$. As Y_t is closed, no vertex of Y_t points to y. Note that $R_{m,H}(\mathcal{C})[f_t(S_t)] \in \mathcal{C}$. Extend f_t to f_{t+1} by mapping x to y, and let $X_{t+1} = X_t \cup \{x\}$ and $Y_{t+1} = Y_t \cup \{y\}$. It is straightforward to see that f_{t+1} is an isomorphism, and that the sets X_{t+1} and Y_{t+1} are closed.

The case $t + 1$ is even is similarly proven by going back, and so is omitted. We therefore have that the union of the chain of partial isomorphisms $(f_t : t \in \mathbb{N})$

$$F = \bigcup_{t \to \infty} f_t$$

is an isomorphism of G with $R_{m,H}(\mathcal{C})$. □

Analogous to the situation for R and all countable graphs, the graph $R_{m,H}(\mathcal{C})$ has the following universal property.

Corollary 2. *If G is a countable semi-directed graph with initial graph H and constant out-degree m, so that each out-neighbour set of a vertex of G induces a subgraph in $\mathcal{C}$, then $G \leq R_{m,H}(\mathcal{C})$.*

Proof. Let H' be the initial copy of H in G, and let f_0 isomorphically map H' in G to H at time 0 in $R_{m,H}(\mathcal{C})$. As in the proof of Theorem 1, let $(x_t : t \in \mathbb{N}^+)$ be a topological sort of $V(G)\backslash V(H')$.

For $t \geq 0$, suppose that f_t is an isomorphism with domain $X_t = V(H') \cup \{x_1, \ldots, x_t\}$ whose range equals the set Y_t of vertices in $R_{m,H}(\mathcal{C})$. Consider the vertex x_{t+1}. The set of m out-neighbours S_t of x_{t+1} are in X_t, by the properties of the ordering $(x_t : t \in \mathbb{N}^+)$. By hypothesis, S_t induces a subgraph in $\mathcal{C}$. It follows that in $R_{m,H}(\mathcal{C})$, there is a vertex y_{t+1} not in Y_t whose out-neighbours are exactly $f_t(S_t)$. We extend f_t to the isomorphism f_{t+1} which maps x_{t+1} to y_{t+1}.

Define
$$F = \bigcup_{t\to\infty} f_t.$$
Then F witnesses that $G \leq R_{m,H}(\mathcal{C})$. □

2.1. **A random graph process.** We next introduce a random graph process which we name the *Age Dependent Process* (*ADP*). The parameters of the process are m, $\mathcal{C}$, and $H \in \mathcal{C}$. Start with $G_0 \cong H$ with vertices labelled $v_1, \ldots v_m$. For $t \geq 1$ fixed, assume that a digraph G_{t-1} has been defined and there are finitely many vertices in G_{t-1}. At time t, add a new vertex v_{m+t}, and choose a set S of m distinct vertices from $V(G_{t-1})$ so that S induces a subdigraph of $\mathcal{C}$, where a vertex v_i is included in the set independently with probability exponentially proportional to the time it was born. More precisely, denote

$$\begin{aligned} L_{t-1} &= \{(j_1, \ldots, j_m) \in \mathbb{N}^m : G_{t-1}[\{v_{j_1}, \ldots, v_{j_m}\}] \in \mathcal{C}, \\ &\qquad v_{j_1}, \ldots, v_{j_m} \in V(G_{t-1}) \text{ are distinct}\}. \end{aligned}$$

For each $S = \{v_{i_1}, \ldots, v_{i_m}\}$ where $(i_1, \ldots, i_m) \in L_{t-1}$, define
$$\mu(S) = 2^{-(i_1+\cdots+i_m)}$$
and
$$N_t = \sum_{(j_1,\ldots,j_m)\in L_{t-1}} 2^{-(j_1+j_2+\cdots+j_m)}.$$
In particular, N_t is the sum of all the $\mu(S)$, where S is a subset of cardinality m from $V(G_{t-1})$ such that $G_{t-1}[S] \in \mathcal{C}$. The probability that S is chosen from $V(G_{t-1})$ equals $\mu(S)/N_t$; this clearly defines a probability measure on m-subsets S with $G_t[S] \in \mathcal{C}$ in G_t. If S is so chosen, then add directed edges from v_{m+t} to each vertex of S.

Theorem 3. *Let $G = \lim_{t\to\infty} G_t$, where G_t is generated by ADP with parameters m, H, and $\mathcal{C}$. Then with probability 1, G is $(\mathcal{C}, m)$-e.c.*

Proof. Fix disjoint finite subsets A and B of $V(G)$ so that $|A| = m$ and $G[A] \in \mathcal{C}$. Let $A = \{v_{i_1}, \ldots, v_{i_m}\}$, where the vertex v_{i_j} was born before $v_{i_{j+1}}$ for all j. Let t_0 be an integer greater than the height of $A \cup B$. For each $t \geq t_0$, let V_t be the event that v_t is pointing to exactly all vertices in A. Note that v_t has out-degree m when it is born, so that if V_t occurs, then there are no edges between v_t and any vertex of B. Then the probability that V_t occurs, written $\mathbb{P}(V_t)$, equals $2^{-(i_1+\cdots+i_m)}/N_t$, where N_t is the normalizing factor defined above.

Note that

$$\begin{aligned} N_t &\leq \sum_{1\leq j_1<j_2<\cdots<j_m\leq t+m-1} 2^{-(j_1+j_2+\cdots+j_m)} \\ &\leq \left(\sum_{j=1}^{t+m-1} 2^{-j}\right)^m \\ &\leq 1, \end{aligned}$$

for all t. Therefore, for all $t \geq t_0$,

$$\mathbb{P}(V_t) \geq 2^{-(i_1+\cdots+i_m)} \geq 2^{-mt_0}.$$

Hence, the probability that there exists no vertex in G that is joined to all vertices in A and none of B is at most

$$\begin{aligned} \mathbb{P}\left(\bigcap_{t=t_0}^{\infty} \overline{V_t}\right) &= \prod_{t=t_0}^{\infty}(1-\mathbb{P}(V_t)) \\ &\leq \lim_{t\geq t_0}(1-2^{-mt_0}) = 0. \end{aligned}$$

As there are only countably many finite subsets A and B and a countable union of measure 0 events is a measure 0 event, the proof follows. □

The following corollary follows immediately from Theorems 1 and 3. It supplies an analogue of the Erdős and Rényi isomorphism result for R.

Corollary 4. *With probability* 1, *a limit graph generated by* ADP *with parameters* m, H, *and* $\mathcal{C}$ *is isomorphic to* $R_{m,H}(\mathcal{C})$.

2.2. **Model-theoretic properties of $R_{m,H}(\mathcal{C})$.** We investigate some of the first-order model-theoretic properties of semi-directed graphs. We work within the *first-order language of graphs* $\mathcal{L}_G$, which contains one binary relation symbol E. We consider a digraph as an $\mathcal{L}_G$-structure, where E is taken as irreflexive binary relation on vertices. The (first-order) $\mathcal{L}_G$*-sentences,* or *sentences,* are defined in the usual way using E, $=$, and the standard logical connectives and quantifiers. An $\mathcal{L}_G$*-theory* is a set of $\mathcal{L}_G$-sentences; the *theory* of an $\mathcal{L}_G$-structure M, written $Th(M)$, is the set of $\mathcal{L}_G$-sentences satisfied by M. For background in first-order logic and its model theory, see [18].

An $\mathcal{L}_G$-theory T is $\aleph_0$*-categorical* if every two countable models of T are isomorphic. If we consider the first-order theory of undirected graphs where E is interpreted as a symmetric binary relation, then $Th(R)$ is $\aleph_0$-categorical, and is *axiomatized* by the e.c. property (that is, the models of $Th(R)$ are exactly the e.c. graphs).

An easy (and so omitted) Compactness argument demonstrates that the on-line property (item 1 in the definition of semi-directedness) is not first-order expressible in $\mathcal{L}_G$. However, for semi-directed graphs we consider the following $\mathcal{L}_G$-theory Φ. The set Φ contains the digraph axioms, and the following sentences. Fix $m > 0$

an integer, and choose $\mathcal{C}$ a good class so it is first-order axiomatizable by finitely many sentences (for example, as is the case for the class of all digraphs). For simplicity, take H to be the digraph with m vertices and no edges.

(1) There are exactly m vertices of out-degree 0 which form an independent set. All other vertices have out-degree m.
(2) The out-neighbour set of each vertex induces a subdigraph in $\mathcal{C}$.
(3) There are no directed cycles.
(4) The $(\mathcal{C}, m)$-e.c. property holds.

It is straightforward to verify that the sentences in Φ are indeed first-order expressible in $\mathcal{L}_G$. We remark that an analogous theory was presented in [19] for the case $m = 2$. By its construction, the digraph $R_{m,H}(\mathcal{C})$ satisfies Φ. However, unlike $Th(R)$, the theory Φ is not $\aleph_0$-categorical. For example, if we let $R_{m,H}(\mathcal{C})_n$ denote the disjoint union of n copies of $R_{m,H}(\mathcal{C})$, where $2 \leq n \leq \aleph_0$, then $R_{m,H}(\mathcal{C})_n$ satisfies Φ and none of these graphs are pairwise isomorphic.

We next observe that $R_{m,H}(\mathcal{C})$ is a *(algebraically) prime model* for Φ: each model of Φ contains an isomorphic copy of $R_{m,H}(\mathcal{C})$. Theorem 5 is an interesting contrast to the universality result of Corollary 2.

Theorem 5. *The digraph $R_{m,H}(\mathcal{C})$ is a prime model for the theory Φ.*

Proof. Suppose that G is a model of Φ. By the $(\mathcal{C}, m)$-e.c. property, H is isomorphic to an induced subdigraph of G. (More explicitly, we may build a copy of H in G inductively, by extending a given independent set X by a vertex joined to nothing in X.) Let f_0 isomorphically map H at time 0 in $R_{m,H}(\mathcal{C})$ to an arbitrary but fixed copy X_0 of H in G.

For $t \geq 0$, suppose that f_t is an isomorphism with domain $R_t = V(R_{m,H}(\mathcal{C}))$ whose range is an induced subdigraph X_t of G. Fix a set of vertices S of R_t that has m vertices and which induces a subdigraph in $\mathcal{C}$; by the construction of $R_{m,H}(\mathcal{C})$ the vertex x_S is joined to each vertex of S. By the $(\mathcal{C}, m)$-e.c. property for G, there is a vertex $y_{f_t(S)}$ whose out-neighbour set is exactly $f_t(S)$, and such that there are no directed edges between $X_t \setminus f_t(S)$ and $y_{f_t(S)}$.

We extend f_t to the isomorphism f_{t+1} which maps x_S to $y_{f_t(S)}$ for all choices of S in R_t. It is straightforward to see this is an isomorphism of R_{t+1} onto an induced subdigraph of G. Define

$$F = \bigcup_{t \to \infty} f_t.$$

Then F witnesses that $R_{m,H}(\mathcal{C}) \leq G$. □

3. The group of $R_{m,H}(\mathcal{C})$

The infinite random graph R possesses a rich group of symmetries. In particular, the graph R is *homogeneous*: isomorphisms between finite induced subgraphs extend to automorphisms. The homogeneous graphs were characterized in [22], while the homogeneous digraphs were characterized in [11]. The graph $R_{m,H}(\mathcal{C})$ is not homogeneous; it is not even vertex-transitive: two vertices with different heights are in different orbits of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$. Hence, the symmetries exhibited by R and $R_{m,H}(\mathcal{C})$ are quite different.

Henson [17] proved that $\mathrm{Aut}(R)$ embeds (that is, contains subgroups isomorphic to) all countable groups. We now prove that the group $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ shares this property with R. Given a set X, we use the notation $\mathrm{Sym}(X)$ for the group of permutations of X. For a set S of vertices and automorphism f, $f(S)$ is the image of S under f.

Theorem 6. *The group* $\mathrm{Sym}(X)$ *embeds in* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$*, where* X *is countably infinite. In particular, each countable group embeds in* $\mathrm{Aut}(R_m(\mathcal{C}))$*.*

Theorem 6 is, after Corollary 2, our second universality result for $R_{m,H}(\mathcal{C})$. Before we prove Theorem 6 we need the following lemma. The graph $R_{m,H}(\mathcal{C})'$ is defined analogously to $R_{m,H}(\mathcal{C})$, but at each time-step R_{t+1}, *infinitely many* vertices x_S are joined to each induced subdigraph of order m from $\mathcal{C}$ in R_t.

Lemma 7. *The digraph* $R_{m,H}(\mathcal{C})'$ *is isomorphic to* $R_{m,H}(\mathcal{C})$*.*

Proof. It is sufficient to prove that $R_{m,H}(\mathcal{C})'$ satisfies the hypotheses of Theorem 1. By its construction, the graph $R_{m,H}(\mathcal{C})'$ is semi-directed with initial graph H and constant out-degree m. Further, each vertex has its out-neighbour set inducing an m-vertex subdigraph in $\mathcal{C}$. To see that $R_{m,H}(\mathcal{C})'$ satisfies the $(\mathcal{C}, m)$-e.c. property, suppose we are given A a set of m-vertices in $R_{m,H}(\mathcal{C})'$ which induces a graph in $\mathcal{C}$, and a finite set B of vertices in $R_{m,H}(\mathcal{C})'$ disjoint from A. Let t_0 be the maximum time a vertex of $A \cup B$ was born. A vertex joined to A and not to B may be found in R_{t_0+1}. □

Proof of Theorem 6. Without loss of generality, by Lemma 7 we will work with $R_{m,H}(\mathcal{C})'$ for the remainder of the proof. By Cayley's theorem, it is sufficient to prove that $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(R_{m,H}(\mathcal{C})')$.

We first observe that $\mathrm{Sym}(X)$ embeds in $\mathrm{Aut}(R_1)$. To see this, label the vertices of $V(R_1)\backslash V(R_0)$ as $X = \{x_i : i \in \mathbb{N}\}$. Fix a bijective mapping $f : X \to X$. Define $F : R_1 \to R_1$ which acts as the identity on H, and otherwise acts as f on X. As the x_i have the same out-neighbours in R_1, it follows that F is an automorphism of R_1. Define $\beta : \mathrm{Sym}(X) \to \mathrm{Aut}(R_1)$ by $\beta(f) = F$. It is straightforward to check that β is an injective group homomorphism.

We next prove that there exists an injective group homomorphism

$$\alpha : \mathrm{Aut}(R_1) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})').$$

Once this is established, then $\alpha\beta : \mathrm{Sym}(X) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$ supplies an embedding of $\mathrm{Sym}(X)$ into the automorphism group $\mathrm{Aut}(R_{m,H}(\mathcal{C})')$, and the assertion will follow.

Fix j an automorphism of R_1. Let $J_1 = j$. For $t \geq 1$, assume that J_t is an automorphism of R_t, and the restriction of J_t to R_1 equals J_1. Let $N^+(z)$ be the set of out-neighbours of a vertex z. Define J_{t+1} by

$$J_{t+1}(z) = \begin{cases} J_t(z) & \text{if } z \in V(R_t); \\ x_{J_t(S)} & \text{if } z = x_S \text{ and } S = N^+(z). \end{cases}$$

From the definition of R_{t+1} and the fact that $J_t \in \mathrm{Aut}(R_t)$, it follows that J_{t+1} is an automorphism of R_{t+1}. Note that J_{t+1} restricted to R_t equals J_t.

The map $J = \bigcup_{t\in\mathbb{N}} J_t$ is an automorphism of $\mathrm{Aut}(R_{m,H}(\mathcal{C})')$. Hence, the function $\alpha : \mathrm{Aut}(R_1) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$ defined by $\alpha(j) = J$ is well-defined. It is straightforward to see that α is injective, and that α preserves the identity automorphism.

Now fix $f, g \in \mathrm{Aut}(R_1)$ and $z \in V(R_H)$. We prove by induction on time t that the vertex z was born that

$$\alpha(fg)(z) = \alpha(f)\alpha(g)(z). \tag{3.1}$$

Equation (3.1) will establish that α is an embedding of groups, and is immediate if $t = 0$. Fix $t \geq 1$. Suppose that z was born at time $t+1$ and so z is of the form x_S, where $S = N^+(z) \subseteq V(R_t)$. Then

$$\begin{aligned} \alpha(fg)(z) &= x_{\alpha(fg)(S)} \\ &= x_{\alpha(f)\alpha(g)(S)} \\ &= \alpha(f)\alpha(g)(z). \end{aligned}$$

The second equality follows since the times that vertices of S were born are all strictly less than $t+1$, and by induction hypothesis. □

The property of extending automorphisms of R_1 to automorphisms of all of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ in the proof of Theorem 6 clearly generalizes to any R_t with $t \geq 0$. In particular, $j \in \mathrm{Aut}(R_t)$ extends to $J \in \mathrm{Aut}(R_{m,H}(\mathcal{C}))$, and the map α_t : $\mathrm{Aut}(R_t) \to \mathrm{Aut}(R_{m,H}(\mathcal{C})')$ defined by $\alpha_t(j) = J$ is an injective group embedding. Although $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ is not homogeneous, we may refer to the above property as *temporal homogeneity* : symmetries of the graphs R_t at time t lift to symmetries of the entire limit graph.

We consider some computational consequences of Theorem 6. We refer the reader to Hodges [18] for any terms not explicitly defined. The *language of groups* $\mathcal{L}$ contains a binary function symbol $\cdot$, a unary function $^{-1}$, and a constant symbol 1. A group is considered as an $\mathcal{L}$-structure in the usual way, and satisfies the group axioms. A *universal sentence* is the smallest class of $\mathcal{L}$-formulas which contains the quantifier-free formulas and is closed under conjunction and disjunction, and adding universal quantifiers at the front. The *universal theory of a group G* is the set of universal $\mathcal{L}$-sentences satisfied by G, and the *universal theory of all groups* is the set of universal sentences satisfied by all groups.

Corollary 8. *The group* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ *does not satisfy any non-trivial group identity. In particular,* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ *generates the variety of all groups.*

Proof. Since every countable group embeds into $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ by Theorem 6, so does the free group on a countable set of generators, written $F(X)$. If there were an equation $s = t$ in $\mathcal{L}$ that is not a consequence of the groups axioms, and satisfied by $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$, then $s = t$ would be satisfied by $F(X)$, which is a contradiction. □

A theory T in the language of groups is *decidable* if there is an effective procedure that, given an arbitrary formula in the of the theory, decides whether the formula is a member of the theory or not.

Corollary 9. *The universal theory of* $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ *is undecidable.*

Proof. We first note that the universal theory of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ equals the universal theory of all groups. This follows since every countable group embeds into $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ by Theorem 6, every universal sentence true in $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ will be true in all countable groups and, by the Löwenheim-Skolem Theorem (see [18]), in all groups.

It is well-known that the universal theory of groups is undecidable. This fact follows this by the existence of a group with an undecidable word problem; see [9, 23]. Hence, the universal theory of $\mathrm{Aut}(R_{m,H}(\mathcal{C}))$ is undecidable. □

References

[1] A. Barabási, R. Albert, Emergence of scaling in random networks, *Science* **286** (1999) 509-512.

[2] B. Bollobás, O. Riordan, J. Spencer, G. Tusnády, The degree sequence of a scale-free random graph process, *Random Structures and Algorithms* **18** (2001) 279-290.

[3] B. Bollobás, O. Riordan, Mathematical results on scale-free graphs, *Handbook of graphs and networks* (S. Bornholdt, H. Schuster eds.), Wiley-VCH, Berlin (2002) 1-32.

[4] A. Bonato, *A Course on the Web Graph*, American Mathematical Society Graduate Studies Series in Mathematics, Providence, Rhode Island, 2008.

[5] A. Bonato, D. Delić, On a problem of Cameron's on inexhaustible graphs, *Combinatorica* **24** (2004) 35-51.

[6] A. Bonato, D. Delić, C. Wang, Universal random semi-directed graphs, In: *Proceedings of ROGICS'08*, 2008.

[7] A. Bonato, J. Janssen, Infinite limits of copying models of the web graph, *Internet Mathematics* **1** (2004) 193-213.

[8] A. Bonato, J. Janssen, Infinite limits and adjacency properties of a generalized copying model, *Internet Mathematics* **4** (2009) 199-223.

[9] W.W. Boone, The word problem, *Annals of Mathematics* **70** (1959) 207-265.

[10] P.J. Cameron, The random graph, In: *The Mathematics of Paul Erdős, II*, Algorithms and Combinatorics, **14**, Springer, Berlin, 1997, pp. 333-351.

[11] G. Cherlin, *The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-Tournaments* , Memoirs of the American Mathematical Society, **131**, Rhode Island, 1998.

[12] F. Chung, L. Lu, T. Dewey, D. Galas, Duplication models for biological networks, *Journal of Computational Biology* **10** (2003) 677-687

[13] F.R.K. Chung, L. Lu, *Complex Networks*, American Mathematical Society, Providence, RI, 2006.

[14] R. Diestel, *Graph theory*, Springer-Verlag, New York, 2000.

[15] P. Erdős, A. Rényi, Asymmetric graphs, *Acta Mathematica Academiae Scientiarum Hungaricae* **14** (1963) 295-315.

[16] R. Fraïssé, *Theory of relations*, *Revised edition, with an appendix by Norbert Sauer*, North-Holland Publishing Co., Amsterdam, 2000.

[17] C.W. Henson, A family of countable homogeneous graphs, *Pacific Journal of Mathematics* **38** (1971) 69-83.

[18] W. Hodges, *Model Theory*, Encyclopedia of Mathematics and its Applications, Vol. 42, Cambridge University Press, Cambridge, 1993.

[19] J. Kleinberg, R.D. Kleinberg, Isomorphism and embedding problems for infinite limits of scale-free graphs, In: *Proceedings of the Sixteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 2005.

[20] R. Kumar, P. Raghavan, S. Rajagopalan, D. Sivakumar, A. Tomkins, E. Upfal, Stochastic models for the web graph, In: *Proceedings of the 41th IEEE Symposium on Foundations of Computer Science*, 2000.

[21] R. Kumar, J. Novak, A. Tomkins, Structure and evolution of online social networks, In: *Proceedings of the 12th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, 2006.

[22] A.H. Lachlan, R.E. Woodrow, Countable ultrahomogeneous undirected graphs, *Transactions of the American Mathematical Society* **262** (1980) 51-94.

[23] P.S. Novikov, On the algorithmic unsolvability of the word problem in group theory, *Trudy Mat. Inst. im. Steklov* **44** (1955) 1-143. (Russian)

[24] D.B. West, *Introduction to Graph Theory, 2nd edition*, Prentice Hall, 2001.

Department of Mathematics, Ryerson University, Toronto, ON, Canada
E-mail address: abonato@ryerson.ca

Department of Mathematics, Ryerson University, Toronto, ON, Canada
E-mail address: ddelic@ryerson.ca

Department of Mathematics, Ryerson University, Toronto, ON, Canada
E-mail address: cpwang@ryerson.ca